$\pi\pi$ and $K\bar{K}$ final-state interactions

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TU Dortmund, 24/04/2024

1. Motivation: pion vector and scalar form factors
2. Dispersion relations and Omnès formalism
3. S-wave and scalar form factors: coupled channels
4. Application: $\bar{B}^0_{d/s} \to J/\psi \pi^+ \pi^-$
5. Going beyond 1 GeV
6. Conclusion and outlook

Motivation

- pion form factors (vector/scalar)
  - describe hadronisation of currents into pairs of pions:

$$\langle \pi^+\pi^- | \tfrac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) | 0 \rangle = F_\pi^V(s)(p_+ - p_-)^\mu$$

$$\langle \pi^+\pi^- | \tfrac{1}{2}(\bar{u}u + \bar{d}d) | 0 \rangle = \mathcal{B}^n \Gamma^n(s) \qquad \langle \pi^+\pi^- | \bar{s}s | 0 \rangle = \mathcal{B}^s \Gamma^s(s)$$

  - strongly constrained model-independently by dispersion theory ($s \lesssim 1$ GeV)

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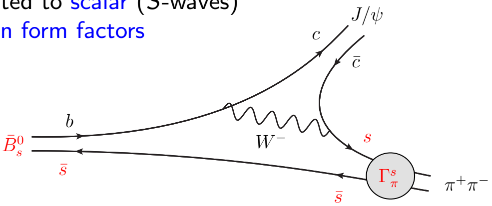
  - ▷ universality of final-state interactions (FSI)
    ⇝ rescattering in $\pi^+\pi^-$ related to scalar (S-waves) and vector (P-waves) pion form factors



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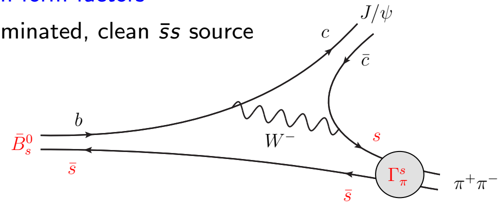

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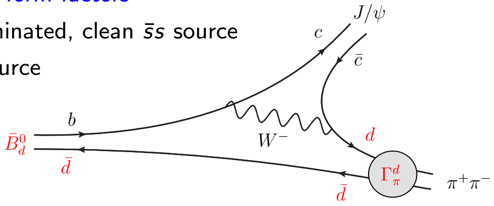

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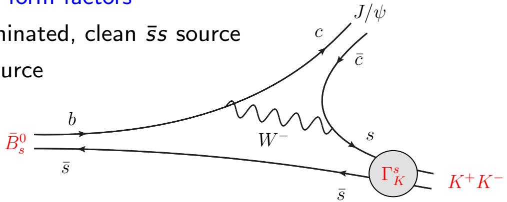
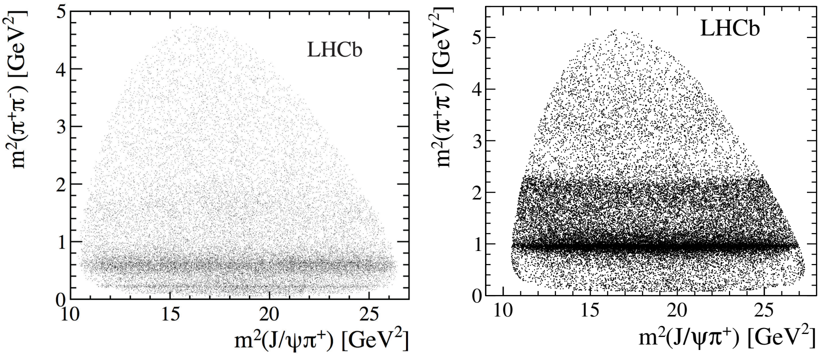
  - ▷ link to $\bar{B}^0_{d/s} \to J/\psi K^+K^-$



Motivation

- no $J/\psi\pi$ structure found by LHCb $\rightsquigarrow$ no left-hand cuts
  - ▷ Dalitz plots:

$\bar{B}_d^0 \to J/\psi\pi^+\pi^-$ $\qquad$ $\bar{B}_s^0 \to J/\psi\pi^+\pi^-$

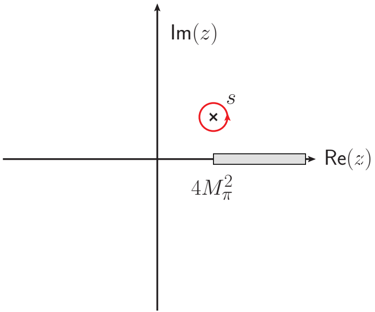


LHCb 2014

- close-to-zero $J/\psi\pi$ scattering length

Liu et al. 2008

Dispersion relations for pedestrians



analyticity ($\simeq$ causality)

& Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

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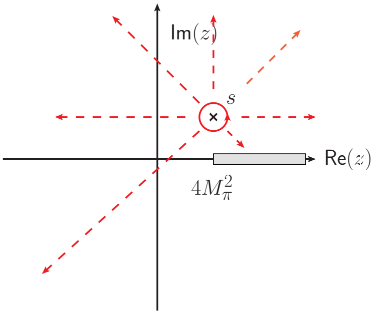

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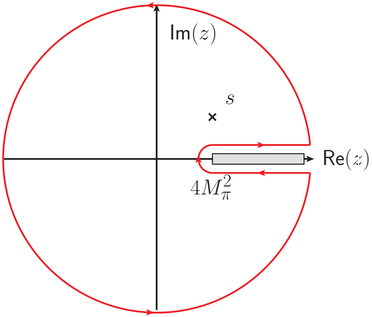

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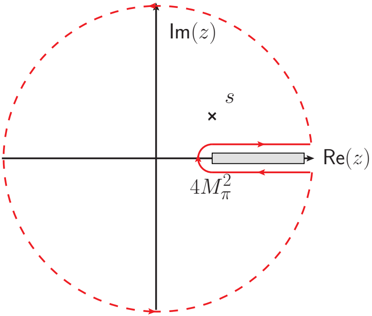

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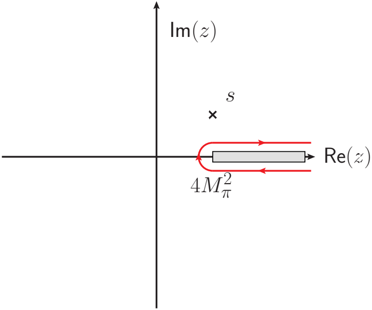

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$$T(s) = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{T(z)dz}{z-s}$$
$$\longrightarrow \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty}\frac{\text{disc}\,T(z)dz}{z-s}$$
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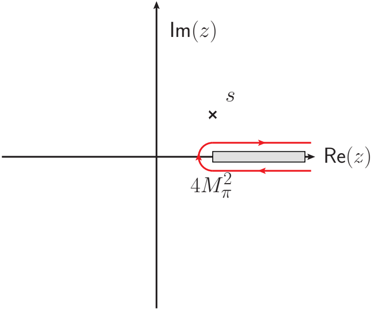

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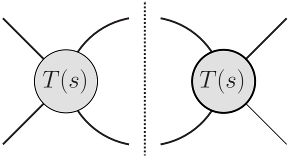
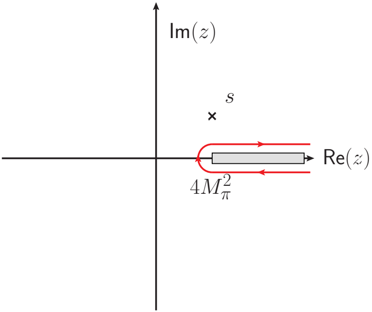
- disc $T(s) = 2i\,\text{Im}\,T(s)$ given by unitarity ($\simeq$ prob. conservation):



e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\text{disc}\,T(s)}{2i} = \text{Im}\,T(s) = \frac{2q_\pi}{\sqrt{s}}\theta(s-4M_\pi^2)|T(s)|^2$$

Dispersion relations for pedestrians



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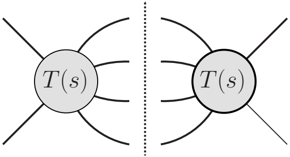
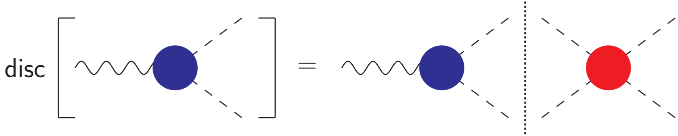


inelastic intermediate states ($K\bar{K}$, 4π)
suppressed at low energies
$\longrightarrow$ important at higher energies

Form factors from *elastic* rescattering

- unitarity relation:



$$\frac{1}{2i}\,\text{disc}\,F_I(s) \;=\; \text{Im}\,F_I(s) \;=\; F_I(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_I(s)\,e^{-i\delta_I(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_I(s)$ is just $\delta_I(s)$ Watson 1954

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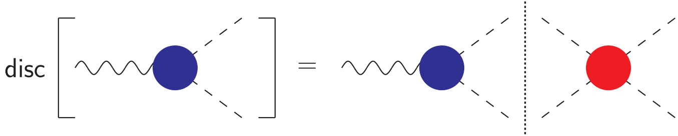

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- solution to this homogeneous integral equation known:

$$F_I(s) = P_I(s)\Omega_I(s)\;, \quad \Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_I(s')}{s'(s'-s)}\right\}$$

$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function Omnès 1958

- today: high-accuracy $\pi\pi$ (and πK) phase shifts available

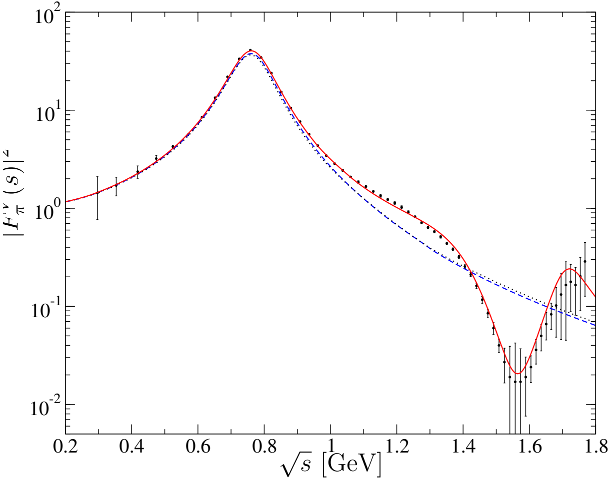
Ananthanarayan et al. 2001, García-Martín et al. 2011

(Büttiker et al. 2004, Peláez, Rodas 2022)

- constrain $P_I(s)$ using symmetries (normalisation at $s=0$ etc.)

Pion vector form factor vs. Omnès representation

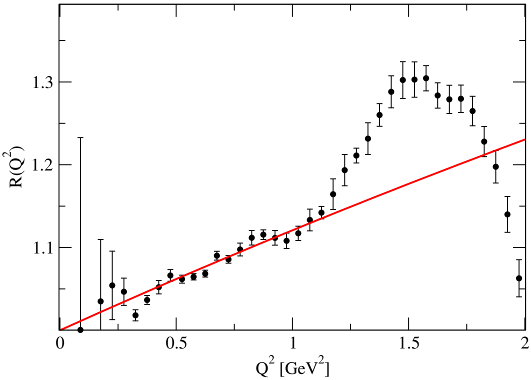
- $\tau^- \to \pi^-\pi^0\nu_\tau$ form factor:



Schneider, BK, Niecknig 2012; data: Belle 2008

Pion vector form factor vs. Omnès representation

- divide $\tau^- \to \pi^-\pi^0\nu_\tau$ form factor by Omnès function:



Hanhart et al. 2013; data: Belle 2008

$\longrightarrow$ linear below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\text{GeV}^{-2}s)\Omega(s)$

slope at $s = 0$ given by elastic contribution to better than 90%

$\longrightarrow$ above: inelastic resonances ρ', ρ''...

Digression: tensor form factors

- interesting for many BSM applications: tensor current form factors

$$\langle \pi^+\pi^- | \bar{q}\sigma^{\mu\nu} q | 0 \rangle = \frac{i}{M_\pi}\left(p_-^\mu p_+^\nu - p_+^\mu p_-^\nu\right) B_T^{\pi,q}(s)$$

- unitarity relation: $\operatorname{Im} B_T^{\pi,q}(s) = \sigma_\pi(s)\left(t_1^1(s)\right)^* B_T^{\pi,q}(s)$

  $\longrightarrow$ identical to the one for $F_V^\pi(s)$ — P-wave form factor!

  $\longrightarrow$ up to inelastic corrections (assuming Brodsky–Lepage asymptotics)

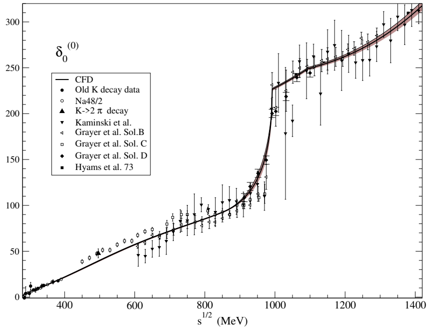
$$B_T^{\pi,q}(s) = B_T^{\pi,q}(0) F_\pi^V(s)$$

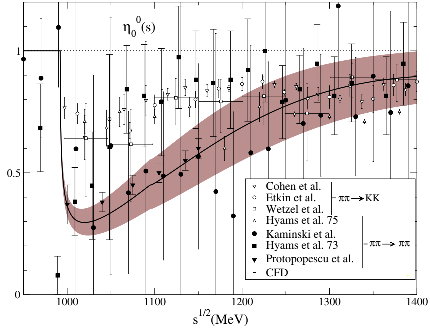
- $B_T^{\pi,u}(0) = -B_T^{\pi,d}(0) = 0.195(10)$ from lattice — Baum et al. 2011
- similar relation for πK tensor form factor — Cirigliano, Crivellin, Hoferichter 2017
- some can even be carried over to nucleon form factors of the tensor current

Hoferichter, BK, Ruiz de Elvira, Stoffer 2018

Pion–pion S-wave: non-Breit–Wigner and $K\bar{K}$ threshold

- isospin $I = 0$ pion–pion S-wave phase and inelasticity:





García-Martín et al. 2011

- phase motion is nowhere near a Breit–Wigner-type shape
- $K\bar{K}$ threshold coincides with $f_0(980)$ resonance

  $\longrightarrow$ strong inelasticity variation, very different from P-wave

  $\longrightarrow$ requires coupled-channel treatment $\pi\pi \leftrightarrow K\bar{K}$

Scalar form factors: coupled channels

- two scalar isoscalar pion form factors:

$$\langle\pi^+\pi^-|\tfrac{1}{2}(\bar{u}u+\bar{d}d)|0\rangle = \mathcal{B}^n\Gamma_\pi^n(s) \qquad \langle\pi^+\pi^-|\bar{s}s|0\rangle = \mathcal{B}^s\Gamma_\pi^s(s)$$

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- two-channel discontinuity equation:

$$\operatorname{disc} \Gamma(s) = 2i\, T^*(s) \Sigma(s) \Gamma(s) \qquad \Gamma(s) = \begin{pmatrix} \Gamma_\pi(s) \\ \frac{2}{\sqrt{3}} \Gamma_K(s) \end{pmatrix}$$

phase space: $\Sigma(s) = \operatorname{diag}\left(\sigma_\pi(s)\theta(s-4M_\pi^2), \sigma_K(s)\theta(s-4M_K^2)\right)$

- parametrisation of two-channel T-matrix:

$$T = \begin{pmatrix} \dfrac{\eta(s)e^{2i\delta(s)}-1}{2i\sigma_\pi(s)} & |g(s)|e^{i\psi(s)} \\ |g(s)|e^{i\psi(s)} & \dfrac{\eta(s)e^{2i(\psi(s)-\delta(s))}-1}{2i\sigma_K(s)} \end{pmatrix}$$

inelasticity: $\quad \eta(s) = \sqrt{1 - 4\sigma_\pi(s)\sigma_K(s)|g(s)|^2\theta(s-4M_K^2)}$

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$\longrightarrow$ three input functions:

- $\triangleright$ $\pi\pi$ S-wave phase shift $\delta(s)$ Caprini, Colangelo, Leutwyler 2012
- $\triangleright$ modulus $|g(s)|$ and phase $\psi(s)$ of $\pi\pi \to K\bar{K}$ amplitude

Büttiker et al. 2004; Cohen et al. 1980, Etkin et al. 1982

Peláez, Rodas 2022

- solution in terms of Omnès matrix

$$\begin{pmatrix} \Gamma_\pi(s) \\ \frac{2}{\sqrt{3}}\Gamma_K(s) \end{pmatrix} = \begin{pmatrix} \Omega_{11}(s) & \Omega_{12}(s) \\ \Omega_{21}(s) & \Omega_{22}(s) \end{pmatrix} \begin{pmatrix} \Gamma_\pi(0) \\ \frac{2}{\sqrt{3}}\Gamma_K(0) \end{pmatrix}$$

Donoghue, Gasser, Leutwyler 1990

Scalar form factors: numerical results

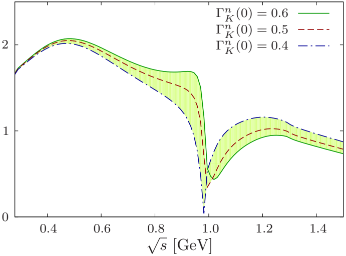
- different scalar form factors depend on normalisation at $s = 0$:

$$\left\langle \pi^+\pi^- \middle| \tfrac{1}{2}(\bar{u}u + \bar{d}d) \middle| 0 \right\rangle = \mathcal{B}^n \Gamma_\pi^n(s) \qquad \left\langle \pi^+\pi^- \middle| \bar{s}s \middle| 0 \right\rangle = \mathcal{B}^s \Gamma_\pi^s(s)$$

- normalisation fixed by Feynman–Hellmann theorem and ChPT:

$$\Gamma_\pi^n(0) = 0.98,\ \Gamma_K^n(0) = \{0.4 \ldots 0.6\},\ \Gamma_\pi^s(0) = 0,\ \Gamma_K^s(0) = \{0.95 \ldots 1.15\}$$

$|\Gamma_\pi^n(\sqrt{s})|$



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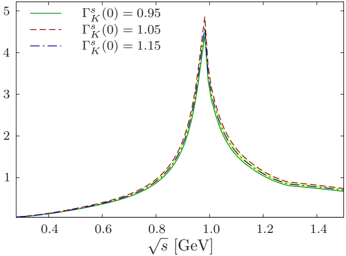


- broad bump: $f_0(500)$ / "σ"
- dip near $f_0(980)$ pole
- prominent peak of the $f_0(980)$

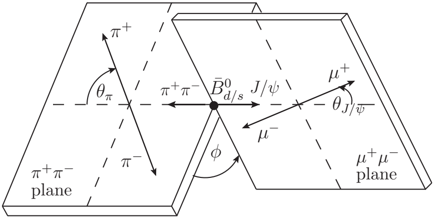
Daub, Hanhart, BK 2015

$\bar{B}^0_{d/s} \to J/\psi\pi^+\pi^-$

- matrix element:

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}} V_{cb}V_{cd}^* f_\psi M_\psi \epsilon_\mu^*(p_\psi,\lambda)\left(\frac{2M_\psi P^\mu_{(0)}}{\lambda^{1/2}(s,m_\psi^2,m_B^2)}\mathcal{F}_0 + \frac{Q^\mu_{(\|)}}{\sqrt{s}}\mathcal{F}_\| - \frac{i\bar{p}^\mu_\perp}{\sqrt{s}}\mathcal{F}_\perp\right)$$

- $\mathcal{F}_{0,\|,\perp}(s,\theta_\pi)$ transversity form factors
  $\longrightarrow$ orthogonal basis of momentum vectors $P^\mu_{(0)}$, $Q^\mu_{(\|)}$, $\bar{p}^\mu_\perp$ Faller et al. 2014



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- $\mathcal{F}_{0,\parallel,\perp}(s, \theta_\pi)$ transversity form factors

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- expand in partial waves f^ℓ_τ $\rightsquigarrow$ Omnès formalism:

$$f^{(S)}_0(s) = b^n_0(1 + b'^n_0 s)\Gamma^n_\pi(s) + c^s_0 \Gamma^s_\pi(s)$$

$$f^{(P)}_\tau(s) = a_\tau(1 + a'_\tau s)\Omega^1_1(s)\left(1 + \frac{\kappa\, s}{M^2_\omega - iM_\omega\Gamma_\omega - s}\right)$$

  - ▷ adjust normalisations, potentially allow for slope parameters
  - ▷ ρ–ω mixing strength κ fixed

$\bar{B}^0_{d/s} \to J/\psi\pi^+\pi^-$

- matrix element:

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}} V_{cb} V^*_{cd} f_\psi M_\psi \epsilon^*_\mu(p_\psi, \lambda) \left(\frac{2M_\psi P^\mu_{(0)}}{\lambda^{1/2}(s, m^2_\psi, m^2_B)} \mathcal{F}_0 + \frac{Q^\mu_{(\|)}}{\sqrt{s}} \mathcal{F}_\| - \frac{i\bar{p}^\mu_\perp}{\sqrt{s}} \mathcal{F}_\perp \right)$$

- $\mathcal{F}_{0,\|,\perp}(s, \theta_\pi)$ transversity form factors
  $\longrightarrow$ orthogonal basis of momentum vectors $P^\mu_{(0)}$, $Q^\mu_{(\|)}$, $\bar{p}^\mu_\perp$ Faller et al. 2014
- expand in partial waves f^ℓ_τ $\rightsquigarrow$ Omnès formalism:

$$f^{(S)}_0(s) = b^n_0(1 + b'^n_0 s)\Gamma^n_\pi(s) + c^s_0 \Gamma^s_\pi(s)$$

$$f^{(P)}_\tau(s) = a_\tau(1 + a'_\tau s)\Omega^1_1(s)\left(1 + \frac{\kappa\, s}{M^2_\omega - iM_\omega\Gamma_\omega - s}\right)$$

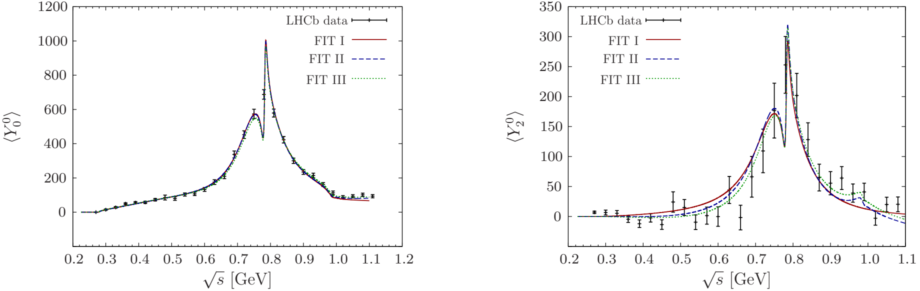
  - ▷ adjust normalisations, potentially allow for slope parameters
  - ▷ ρ–ω mixing strength κ fixed
- comparison to data: fit to angular moments LHCb 2014

$$\langle Y^0_l \rangle(s) = \int_{-1}^{1} \frac{d^2\Gamma}{d\sqrt{s}\, d\cos\theta_\pi} Y^0_l(\cos\theta_\pi) d\cos\theta_\pi$$

$\langle Y^0_0 \rangle \propto d\Gamma/d\sqrt{s}$, $\langle Y^0_2 \rangle$: P-waves, D-waves, S–D-interference

$\bar{B}_d^0 \to J/\psi\pi^+\pi^-$: fit results, S-wave

- FIT I: 3 parameters (b_0^n, a_0, $a_{||}$); FIT II: + D-wave; FIT III: + $a_0' \neq 0$ (4 par.)



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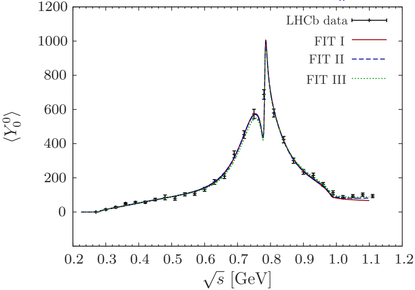
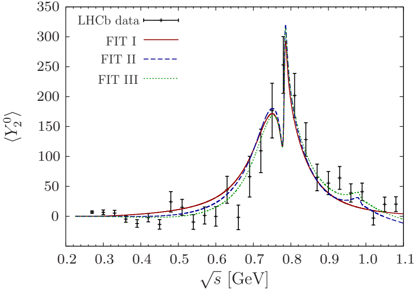
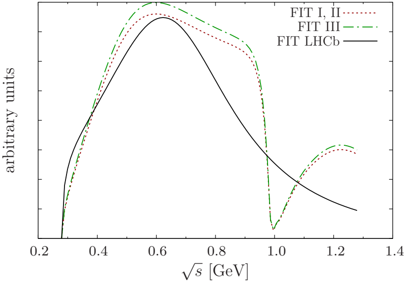
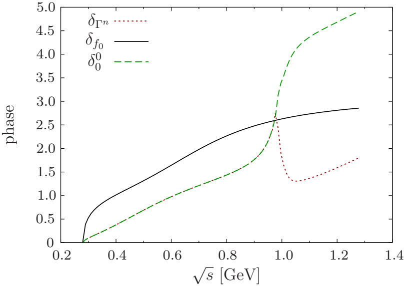


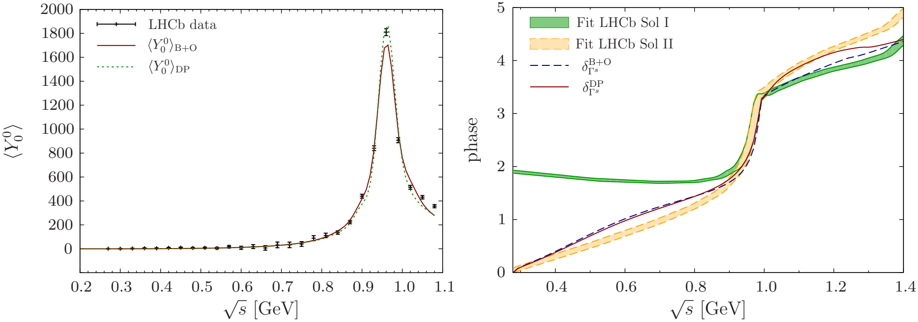


- drastic differences in extracted S-wave modulus and phase vs. LHCb:





$\bar{B}_s^0 \to J/\psi\pi^+\pi^-$: fit results, S-wave



- phase behaviour: dispersive constraints select "correct" LHCb solution

Daub, Hanhart, BK 2015

Going beyond 1 GeV: higher states and resonances

- $\pi\pi$ and $K\bar{K}$ coupled channels work up to $1.1\,\mathrm{GeV}$
- beyond: strong coupling to 4π $\longrightarrow$ phase/inelasticity description??
- resonances, e.g. $\mathcal{B}(f_0(1500) \to 4\pi) = (49.5 \pm 3.3)\%$ PDG 2018

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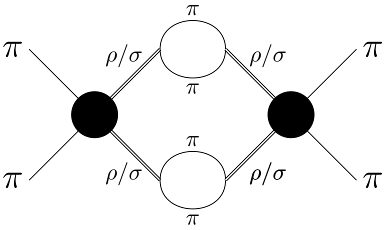
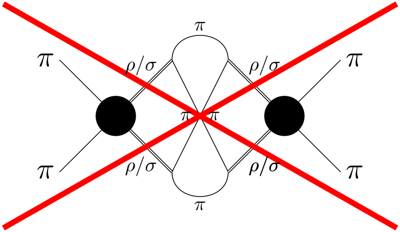
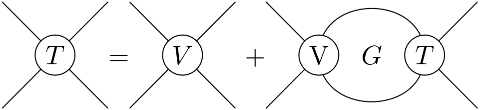
  $\longrightarrow$ Omnès at low energies, unitary isobar model above
- 4π in general very complicated; approximations:
  - vector form factor: 4π phase space only $+\ \pi\omega$ Hanhart 2012
  - scalar form factor: isobars $\rho\rho$ or $\sigma\sigma$ Ropertz, Hanhart, BK 2018
- neglect crossed-channel effects, other channels



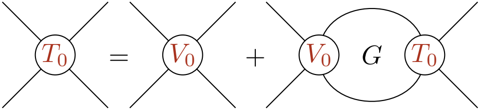


Partial-wave amplitude: 2-potential formalism

- Bethe–Salpeter equation for partial-wave amplitude T:

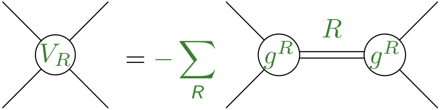


- split scattering kernel $V = V_0 + V_R \longrightarrow T = T_0 + T_R$
- unitary scattering amplitude T_0 (given by known phases and inelasticities)



$$= \begin{pmatrix} \frac{\eta e^{2i\delta}-1}{2i\sigma_\pi} & g e^{i\psi} & 0 \\ g e^{i\psi} & \frac{\eta e^{2i(\psi-\delta)}}{2i\sigma_K} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- resonance-exchange potential V_R



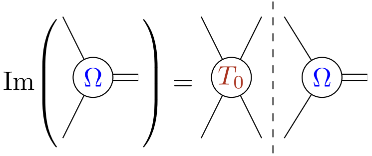
$$= -\sum_R g_i^R \frac{s}{m_R^2\,(s - m_R^2)} g_j^R$$

Partial-wave amplitude: 2-potential formalism

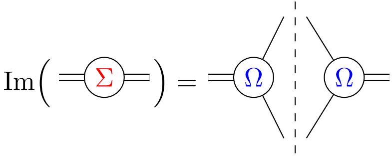
- full parametrisation for scattering matrix T:

$$T = T_0 + \Omega\, [\mathbb{1} - V_R\, \Sigma]^{-1}\ V_R\, \Omega^t$$

vertex factor $\Omega(s)$



self energy $\Sigma(s)$



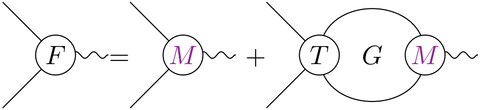
$$\Omega_{ij}(s) = \frac{1}{\pi} \int_{s_{\rm th}}^{\infty} \mathrm{d}z\, \frac{(T_0)^*_{ik}(z)\sigma_k(z)\Omega_{kj}(z)}{z - s - i\epsilon} \qquad \Sigma_{ij}(s) = \frac{s}{\pi} \int_{s_{\rm th}}^{\infty} \frac{\mathrm{d}z}{z}\, \frac{\Omega^*_{ki}(z)\sigma_k(z)\Omega_{kj}(z)}{z - s - i\epsilon}$$

- additional channels: $(T_0)_{ij} = 0 \longrightarrow$

$$\Omega_{ij}(s) = \delta_{ij} \quad \text{and} \quad \Sigma_{ij}(s) = \delta_{ij}\, \frac{s}{\pi} \int_{s_{\rm th}}^{\infty} \frac{\mathrm{d}z}{z}\, \frac{\sigma_i(z)}{z - s - i\epsilon}$$

Form factor parametrisation

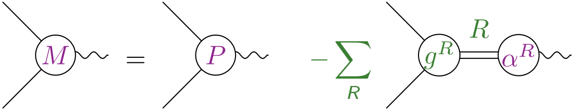
- coupling to a source/current:



- full parametrisation for form factor F:

$$F = \Omega\,[\mathbb{1} - V_R\,\Sigma]^{-1}\,M$$

- source term $M(s)$:



$$M_i(s) = a_i + b_i\,s + \ldots - \sum_R g_i^R \frac{s}{s - m_R^2}\,\alpha^R$$

$\longrightarrow$ new parameters: resonance masses (m_R),
resonance–source (α^R) and resonance–channel (g_i^R) couplings

Application to $\bar{B}_s^0 \to J/\psi\pi^+\pi^-/K^+K^-$

- fit to LHCb angular moments in full energy range:

$$\left\langle Y_L^0 \right\rangle \left(\sqrt{s}\right) = \int_{-1}^{1} \mathrm{d}\cos\Theta \, \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\sqrt{s}\,\mathrm{d}\cos\Theta} \, Y_L^0\left(\cos\Theta\right)$$

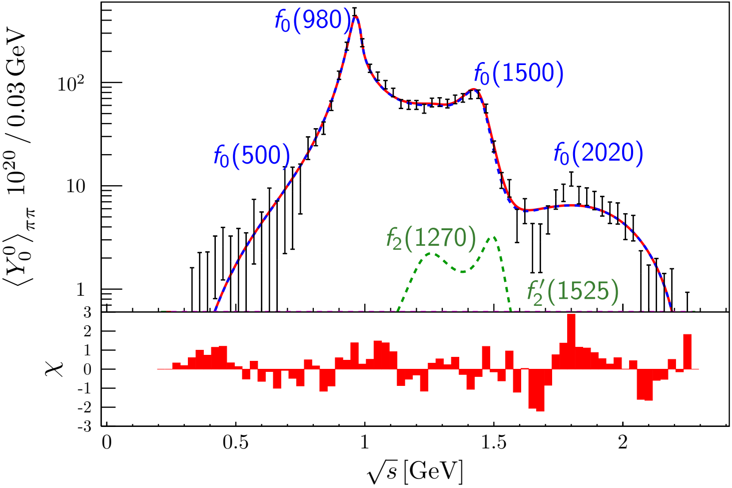
- normalisation $\mathcal{N}$, scalar form factor F_S,
  P-waves F_P^τ (K^+K^- only!), D-waves F_D^τ:

$$\sqrt{4\pi}\left\langle Y_0^0 \right\rangle = X\sigma_\pi\sqrt{s}\left\{X^2\mathcal{N}^2|F_S|^2 + \sum_{\tau=0,\|,\perp}|F_P^\tau|^2 + \sum_{\tau=0,\|,\perp}|F_D^\tau|^2\right\}$$

$$\sqrt{4\pi}\left\langle Y_2^0 \right\rangle = X\sigma_\pi\sqrt{s}\left\{2X\mathcal{N}\mathrm{Re}\left(F_S\left(F_D^0\right)^*\right) + \sum_{\tau=0,\perp,\|}\left(c_\tau\,|F_P^\tau|^2 + d_\tau\,|F_D^\tau|^2\right)\right\}$$

- P- and D-wave amplitudes modelled by Breit–Wigner functions

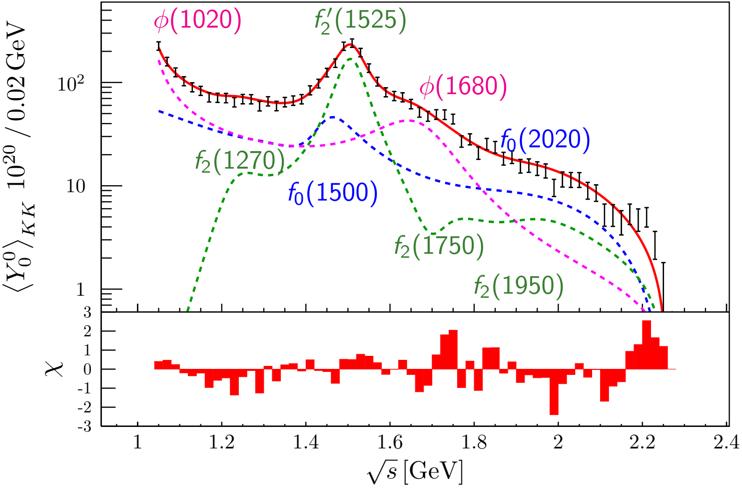
Fit $\langle Y_0^0 \rangle$ for $\bar{B}_s^0 \to J/\psi\, \pi^+\pi^-$



Fit ($\rho\rho$) with two additional resonances. *S*- and *D*-waves

$$\frac{\chi^2}{\text{ndf}} = \frac{376.2}{384 - 30} \approx 1.07$$

Fit $\langle Y_0^0 \rangle$ for $\bar{B}_s^0 \to J/\psi K^+ K^-$



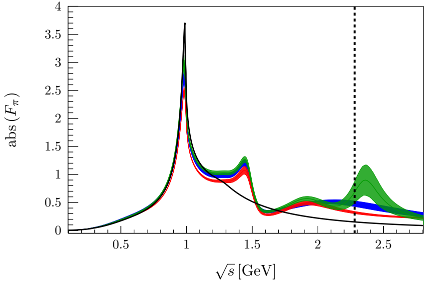
Fit $(\rho\rho)$ with two additional resonances. S-, P-, and D-waves

$$\frac{\chi^2}{\text{ndf}} = \frac{376.2}{384 - 30} \approx 1.07$$

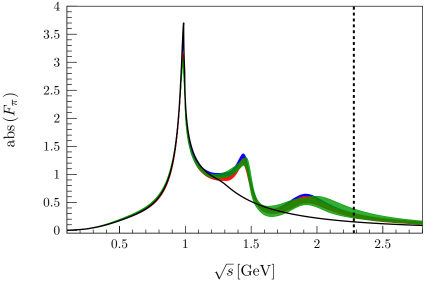
Comparison F_π

$\frac{\chi^2}{\mathrm{ndf}}$	$\rho\rho$	$\sigma\sigma$
Fit 1=2 resonances, constant polynomial in $M(s)$	1.07	1.22
Fit 2=2 resonances, linear polynomial in $M(s)$	1.03	1.18
Fit 3=3 resonances, constant polynomial in $M(s)$	0.96	1.05

$\rho\rho$



$\sigma\sigma$



Omnès solution with $F_\pi(0) = 0$ and $F_K(0) = 1$ in black

$$F_\pi(s) = \Omega_{11}(s)\,F_\pi(0) + \frac{2}{\sqrt{3}}\,\Omega_{12}(s)\,F_K(0)$$

Phase input: Dai, Pennington 2014

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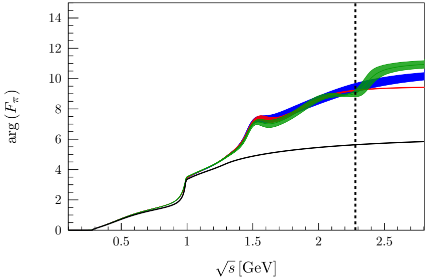

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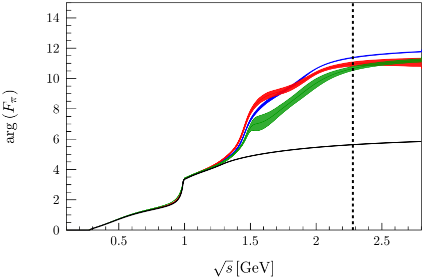

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$$F_\pi(s) = \Omega_{11}(s)\, F_\pi(0) + \frac{2}{\sqrt{3}}\, \Omega_{12}(s)\, F_K(0)$$

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Comparison F_K

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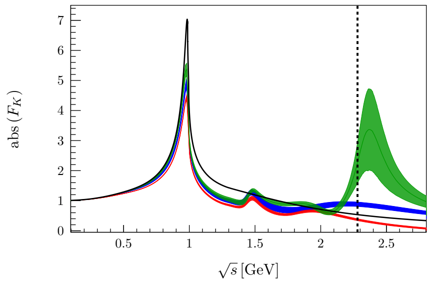
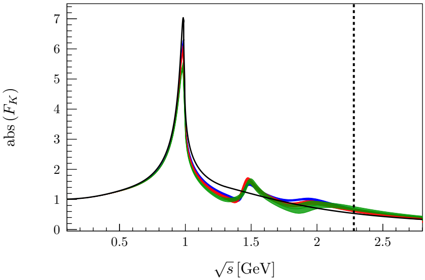
$\rho\rho$



$\sigma\sigma$



Omnès solution with $F_\pi(0) = 0$ and $F_K(0) = 1$ in black

$$\frac{2}{\sqrt{3}} F_K(s) = \Omega_{21}(s)\, F_\pi(0) + \frac{2}{\sqrt{3}} \Omega_{22}(s)\, F_K(0)$$

Phase input: Dai, Pennington 2014

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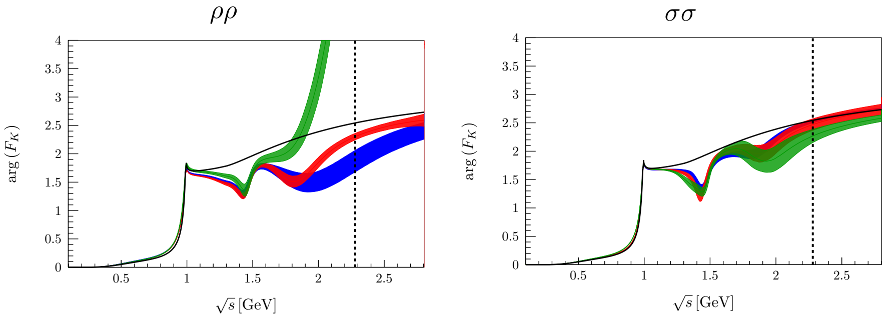

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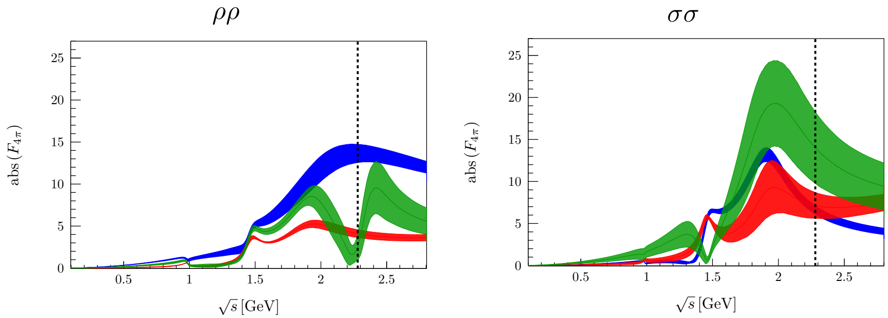
Phase input: Dai, Pennington 2014

Comparison $F_{4\pi}$



Fit 1, Fit 2 and Fit 3

- suppressed at lower energies
- strong model dependence of the additional channel
- **need to include more exclusive data!**

Conclusion and outlook

Conclusion

- elastic unitarity: strongest constraints on vector form factor (tensor, too!)
- coupled channels (scalar $\pi\pi \leftrightarrow K\bar{K}$):
  - requires more scattering input
  - continuous freedom through *relative* channel coupling strength
- modelling the extension to 1–2 GeV:
  - merge low-energy dispersive to unitary isobar model
  - $\bar{B}_s^0 \to J/\psi\pi^+\pi^-/K^+K^- \longrightarrow$ strange scalar form factor
  - not discussed here: extraction of resonance poles

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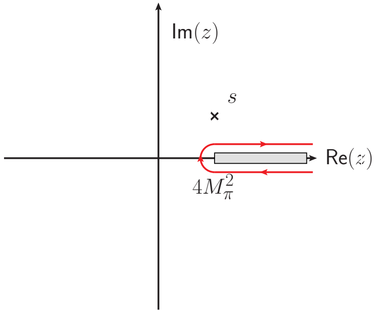
Outlook

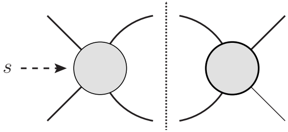
- test universality e.g. in $\bar{B}_s^0 \to \psi'\pi^+\pi^-/K^+K^-$
- similar data for non-strange scalar form factor??
- in progress: coupled-channel treatment of vector form factor $\pi\pi \leftrightarrow \pi\omega$

Chanturia, Heuser et al.

What are left-hand cuts?

Example: pion–pion scattering

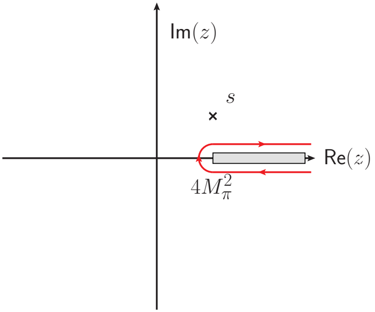


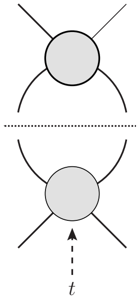


- right-hand cut due to unitarity: $s \geq 4M_\pi$

What are left-hand cuts?

Example: pion–pion scattering





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- crossing symmetry: cuts also for t, $u \geq 4M_\pi$

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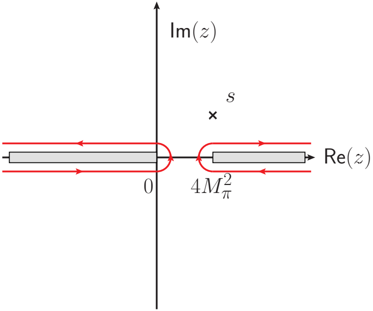
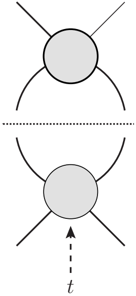




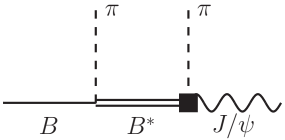
- right-hand cut due to unitarity: $s \geq 4M_\pi$
- crossing symmetry: cuts also for t, $u \geq 4M_\pi$
- partial-wave projection: $T(s,t) = 32\pi \sum_i T_i(s) P_i(\cos\theta)$

$$t(s, \cos\theta) = \frac{1-\cos\theta}{2}(4M_\pi - s)$$

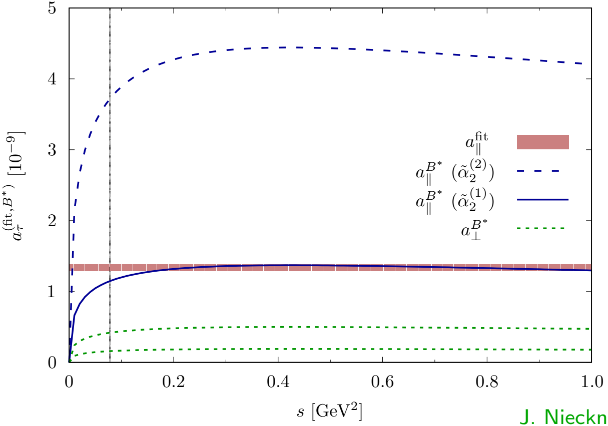
$\longrightarrow$ cut for $t \geq 4M_\pi$ becomes cut for $s \leq 0$ in partial wave

Left-hand cut in $\bar{B}_d^0 \to J/\psi\, \pi^+\pi^-$

- B^*-exchange contribution:



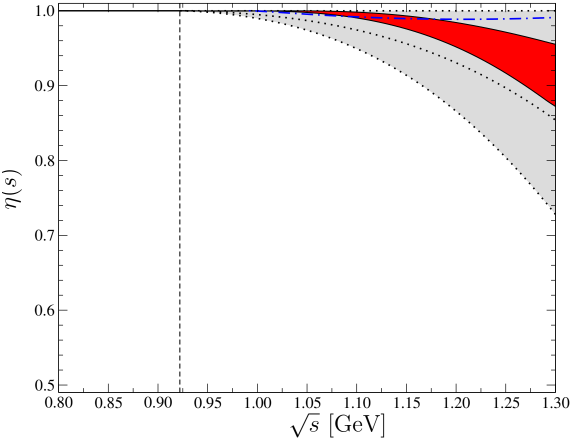
$\longrightarrow$ left-hand cut at $s = 0$; deviation from constant:



J. Niecknig (Daub) 2018

Pion vector form factor: why does this work so well?

- inelastic effects ($\eta(s) \neq 1$) start well above 1 GeV and set in *smoothly*:



grey: phenomenological limits — García-Martín et al. 2011

blue: $K\bar{K}$ — Büttiker et al. 2004

red: $\pi\omega$ — Niecknig, BK, Schneider 2012

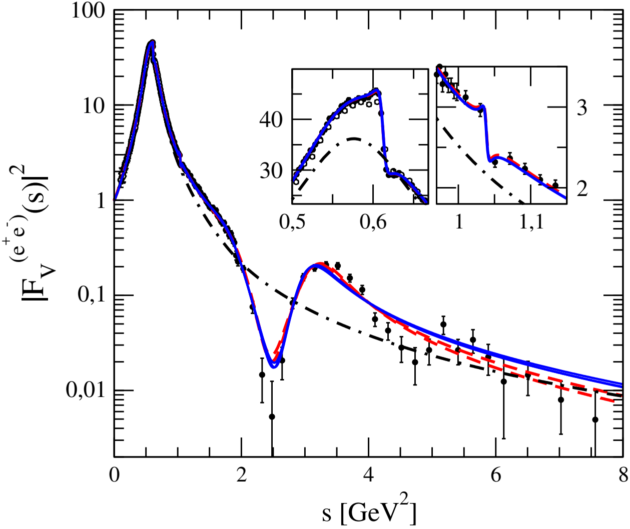
Application to the pion vector form factor

Hanhart 2012

- resonances up to $2\,\mathrm{GeV}$: $\rho(770)$ (elastic!), $\rho(1450)$, $\rho(1700)$
- channels (1–3):
  - ▷ $\pi\pi$ ($\sqrt{s_{th}} \approx 0.29\,\mathrm{GeV}$): elastic, works up to $1\,\mathrm{GeV}$
  - ▷ 4π ($\sqrt{s_{th}} \approx 0.56\,\mathrm{GeV}$): heavily phase-space suppressed at low energies
  - ▷ $\pi\omega$ ($\sqrt{s_{th}} \approx 0.92\,\mathrm{GeV}$): strong role in $\pi\pi$ inelasticity
- elastic scattering matrix

$$T^0(s) = \begin{pmatrix} \frac{\sin(\delta(s))}{\sigma_\pi(s)} e^{i\delta(s)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

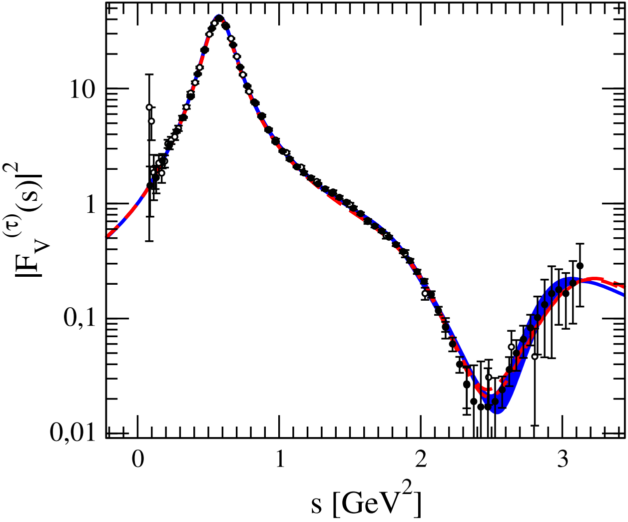
Application to the pion vector form factor



- blue solid: $\pi\pi + 4\pi$ $\left(\frac{\chi^2}{\text{d.o.f.}} = 1.2\right)$
- red dashed: $\pi\pi + 4\pi + \pi\omega$ $\left(\frac{\chi^2}{\text{d.o.f.}} = 1.4\right)$
- data: BaBar 2009, KLOE 2011

Hanhart 2012

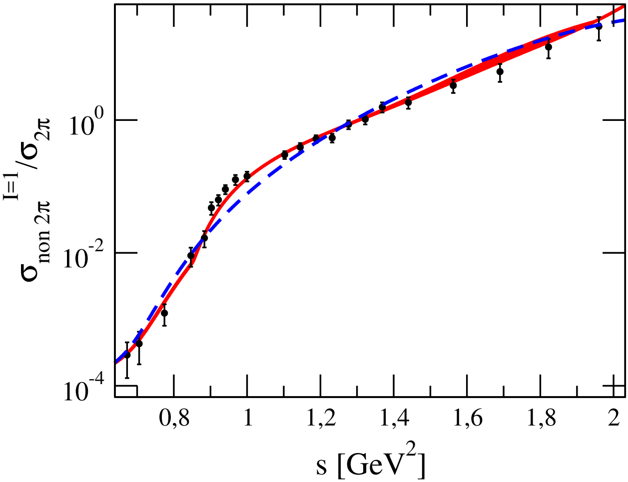
Application to the pion vector form factor



- blue solid: $\pi\pi + 4\pi$ (prediction)
- red dashed: $\pi\pi + 4\pi + \pi\omega$ (prediction)
- data: CLEO 2000, Belle 2008

Hanhart 2012

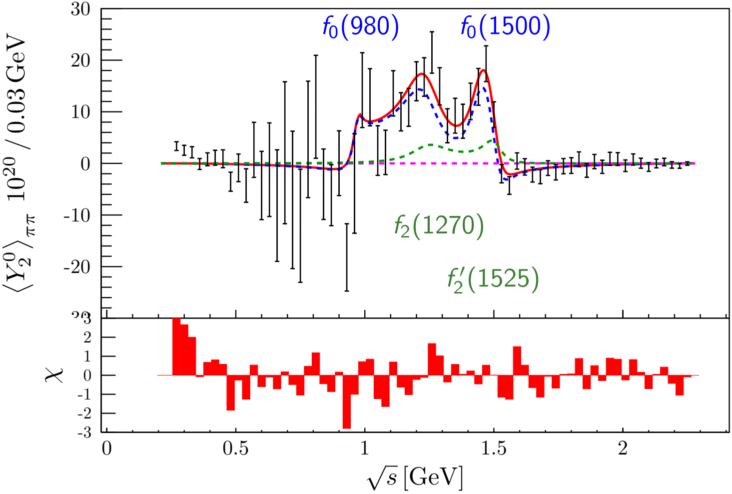
Cross section ratio inelastic vs. elastic



- blue dashed: $\pi\pi$ + 4π
- red solid: $\pi\pi$ + 4π + $\pi\omega$
- data: Eidelman, Łukaszuk 2004

Hanhart 2012

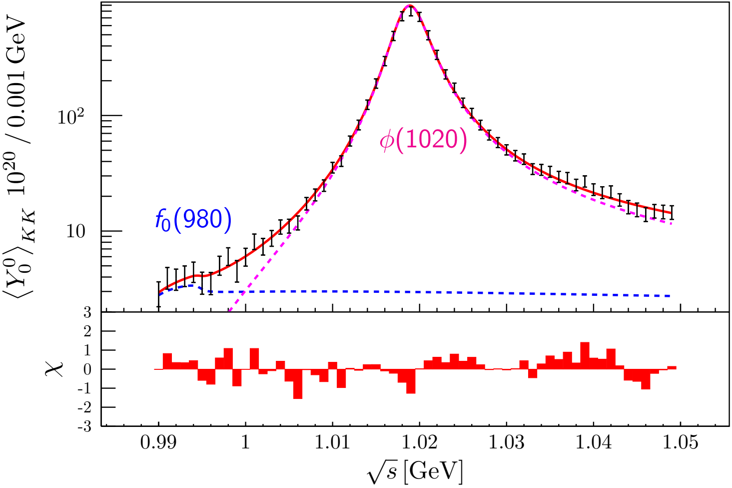
Fit $\langle Y_2^0 \rangle$ for $\bar{B}_s^0 \to J/\psi\, \pi^+\pi^-$



Fit $(\rho\rho)$ with two additional resonances. *SD*- and *D*-waves

$$\frac{\chi^2}{\mathrm{ndf}} = \frac{376.2}{384 - 30} \approx 1.07$$

Fit $\langle Y_0^0 \rangle$ for $\bar{B}_s^0 \to J/\psi\, K^+ K^-$



Fit $(\rho\rho)$ with two additional resonances. S-, P-, and D-waves

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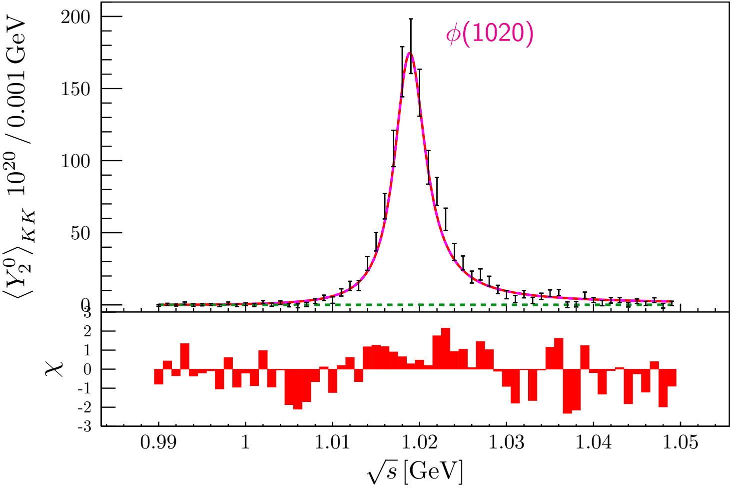

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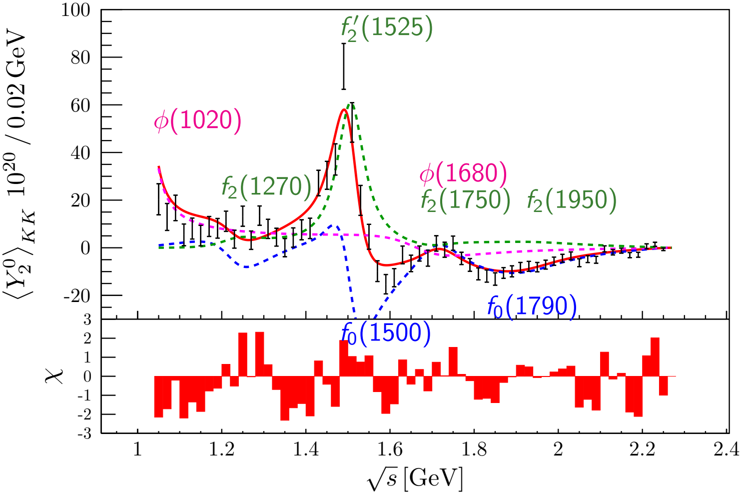

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