The reachability problem for Petri nets

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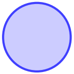
What is a Petri net?

The set of places



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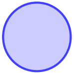
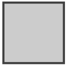


The set of transitions



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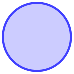
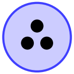


Places store tokens



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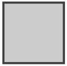

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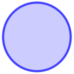

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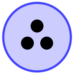

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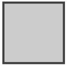
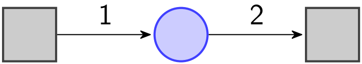
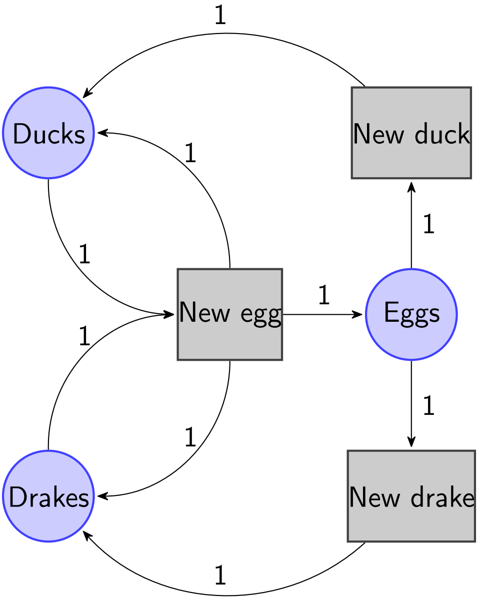


There are arcs between places and transitions labeled with natural numbers.



Duck life simulation



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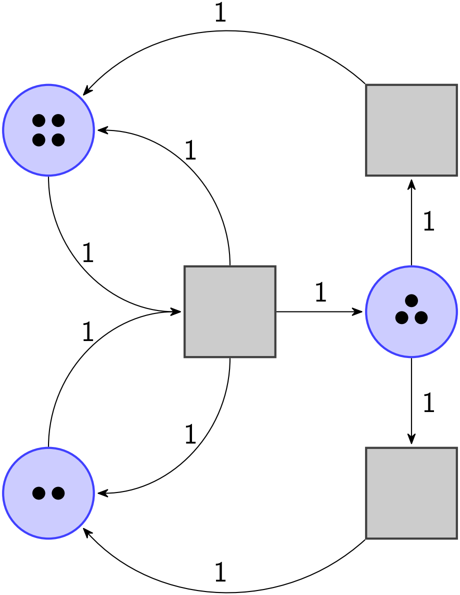

The reachability problem

Configuration = distribution of tokens over the places

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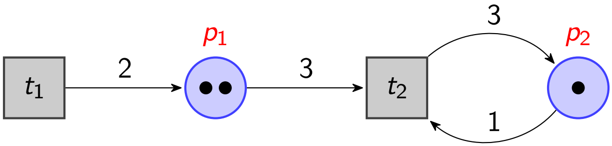
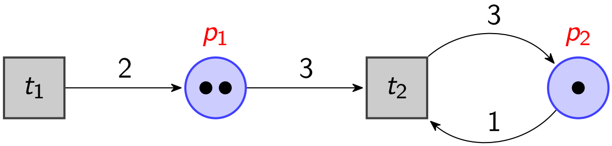
Problem

Given a Petri net and its two configuraions decide if one is reachable from the another.

Example



Can we reach 5 tokens in p_1 and 4 tokens in p_2 from the above configuration?



Can we reach 5 tokens in p_1 and 4 tokens in p_2 from the above configuration?

No! The number of tokens in p_2 is always odd.

Theorem

The reachability problem for Petri nets is Ackermann-complete.

$A_1(n) = 2n$

Fast growing functions and induced complexity classes

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$$A_{i+1}(n) = \underbrace{A_i \circ \cdots \circ A_i}_{n}(1)$$

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$$\mathcal{F}_2 = DTIME(2^{O(n)})$$
$$\mathcal{F}_3 = TOWER$$
$$\ldots$$
$$\mathcal{F}_\omega = ACKERMANN$$

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Open problems

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What is the exact complexity of the reachability problem when we fix the number of places?

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Problem

Is the reachability problem decidable for Petri nets with data?

References

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Find out more

Presentation by Sławomir Lasota about the reachability problem:
`https://www.mimuw.edu.pl/~sl/SLIDES/2023-09-ACPN.pdf`