Exactly solvable Quantum spin-1/2 models in 2D

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Quantum Phase Transition

Exactly solvable system

In Theoretical Physics, in general, we say that a system is exactly solvable, when its behaviour and properties can be fully determined analytically, without any approximations, for all possible conditions and parameters.

The system's Hamiltonian can be diagonalized and its eigenvalues and eigenstates can be explicitely computed.

Ising 1D with transverse field

The model is described by the following Hamiltonian operator:

$$H = -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{N} h \sigma_i^z,$$

where h is an external transverse magnetic field, σ_i^z is a spin of i-th fermion in $\hat{z}$ direction. $\sigma_i^z = 1$, when a spin of i-th fermion is pointing along $\hat{z}$ and $\sigma_i^z = -1$, when a spin of the i-th fermion is pointing in the opposite direction to $\hat{z}$.

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The model can be solved analytically by mapping to the free fermions through Jordan-Wigner transformation:

$$\frac{1}{2}\left(\sigma_j^x + i\sigma_j^y\right) = \sigma_j^+ = \prod_{k=1}^{j-1} \left(-\sigma_k^z\right) c_j^\dagger$$

$$\frac{1}{2}\left(\sigma_j^x - i\sigma_j^y\right) = \sigma_j^- = \prod_{k=1}^{j-1} \left(-\sigma_k^z\right) c_j$$

$$\sigma_j^z = 1 - 2c_j^\dagger c_j$$

$$\{c_j^\dagger, c_i\} = \delta_{ij}$$

$$\{\sigma_j^+, \sigma_i^-\} = \delta_{ij}, \text{for } i = j, \text{ but for } i{\neq}j \; \{\sigma_j^+, \sigma_i^-\} \neq \delta_{ij}$$

Ising 1D with transverse field

In J-W fermions basis $\hat{H}$ would have the form:

$$\hat{H} = -J \sum_{i=1}^{N-1} \left(c_i^\dagger c_{i+1} + c_{i+1} c_i + \mathrm{h.c.} \right) + h \sum_{i=1}^{N} \left(2 c_i^\dagger c_i - 1 \right)$$

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Then we can change the basis of our Hilbert space and move to momentum space by Fourier Transform:

$$c_n = \frac{e^{i\pi/4}}{\sqrt{N}} \sum_k c_k e^{ikn}$$

In new basis Hamiltonian would be:

$$\hat{H} = \sum_k 2 \left(h - Jcos(k) \right) \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) - 2Jsin(k) \left(c_k^\dagger c_{-k}^\dagger - c_{-k} c_k \right)$$

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Let's define a fermionic two-component spinor:

$$\hat{\Psi}_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \quad \hat{\Psi}_k^\dagger = \begin{pmatrix} c_k^\dagger, & c_{-k} \end{pmatrix}$$

$$\hat{H}_k = \Psi_k^\dagger \begin{bmatrix} 2\left(h - J\cos k\right) & 2J\sin k \\ 2J\sin k & -2\left(h - J\cos k\right) \end{bmatrix} \Psi_k$$

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The Bogoliubov rotation leads us to diagonal for of $\hat{H}$:

$$H = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - 1/2 \right), \quad \epsilon_k = 2J\sqrt{(h - \cos k)^2 + \sin k^2}$$

1D Ising model with long-range interactions

Let's add to our system long range interactions of spin-spin which decay as $\frac{1}{r^\alpha}$, $\alpha > 1$:

$$H = -\frac{J}{2}\sum_{i\neq j}\frac{\sigma_i^x\sigma_j^x}{|i-j|^\alpha} - \sum_{i=1}^{N} h\sigma_i^z,$$

Our Hamiltonian becomes non-local and because of that J-W transformation would not give us a quadratic $\hat{H}$ in free fermionic operators. The system is no longer exactly solvable. In this case we relly on numerical techniques such as quantum Monte Carlo, or tensor-networks-based methods.

2D Ising model. Onsager's solution

The system's Hamiltonian for 2D Ising model without transverse field:

$$H_{2D} = -J \sum_{<i,j>} \sigma_i \sigma_j$$

Exactly solvable system

In theory of equilibrium statistical mechanics, the canonical partition function $Z(\beta)$ contains all the information needed to recover the thermodynamical properties of a system with fixed number of particles, immersed in a heat bath.

$$Z(\beta) = \sum_{\sigma} e^{-\beta \hat{H}(\sigma)} = \sum_{\sigma} \prod_{<i,j>} e^{\beta J \sigma_i \sigma_j}$$

The operator T is called transfer matrix:

$$T = e^{\beta J \sigma_i \sigma_j}$$

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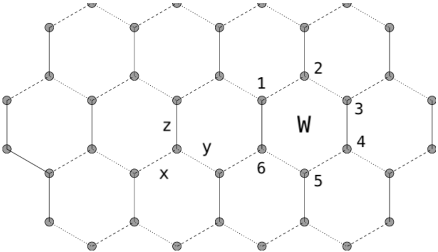
- the largest eigenvalue of T corresponds to the free energy,
- the corresponding eigenvector provides information about the thermodynamic properties of the system, such as the magnetization and correlation functions.

2D Kitaev model

Model has spin-1/2 operators on a 2D honeycomb lattice with isotropic couplings between nearest neighbours. The energy operator is given by:

$$H = \sum_{j+l=\text{even}} \left(J^x \sigma^x_{j,l} \sigma^x_{j+1,l} + J^y \sigma^y_{j,l} \sigma^y_{j-1,l} + J^z \sigma^z_{j,l} \sigma^z_{j,l+1} \right)$$



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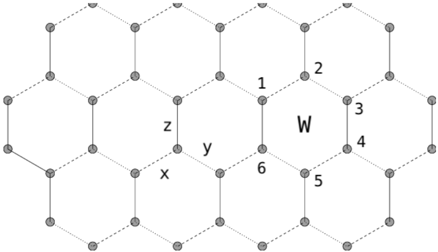

Each plaquette posseses a conserved quantity:

$$W = \sigma^y_1\sigma^z_2\sigma^x_3\sigma^y_4\sigma^z_5\sigma^x_6$$

We can see that W is hermitian, $W^2 = Id$ and all W commute with each other and with H.

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J-W transform is defined as:

$$a_{\vec{n}} = \left[\prod_{\vec{m}=-\infty}^{\vec{n}-\mathbf{1}} \sigma^z_{\vec{m}} \right] \sigma^y_{\vec{n}} (\sigma^x_{\vec{n}}) \quad \text{for even (odd) spins}$$

$$b_{\vec{n}} = \left[\prod_{\vec{m}=-\infty}^{\vec{n}-\mathbf{1}} \sigma^z_{\vec{m}} \right] \sigma^x_{\vec{n}} (\sigma^y_{\vec{n}}) \quad \text{for even (odd) spins}$$

From this it follows that:

$$\sigma^z_{\mathbf{1}} \sigma^z_{\mathbf{2}} \sim a_{\mathbf{1}} b_{\mathbf{2}} W_{\mathbf{1}} W_{\mathbf{2}} W_{\mathbf{3}} ...$$

The J-W transformation gives us quadratic $\hat{H}$ in fermionic operators:

$$H = i \sum_{\vec{n}} \left[J^x_{\vec{n}} b_{\vec{n}} a_{\vec{n}-\vec{M}_{\mathbf{1}}} + J^y_{\vec{n}} b_{\vec{n}} a_{\vec{n}+\vec{M}_{\mathbf{2}}} + J^z_{\vec{n}} b_{\vec{n}} a_{\vec{n}} \right]$$

where:

$$\vec{M}_{\mathbf{1}} = \frac{\sqrt{3}}{2} \hat{j} + \frac{3}{2} \hat{i}, \qquad \vec{M}_{\mathbf{2}} = \frac{\sqrt{3}}{2} \hat{j} - \frac{3}{2} \hat{i}$$

Thank you for your attention!