Geometric decomposition method in solving equations of physics and geometry

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Radosław A. Kycia
kycia.radoslaw@gmail.com
joint work with Josef Šilhan

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Faculty of Computer Science,
T. Kościuszko Cracow University of Technology, Kraków
AND
Department of Mathematics and Statistics
Masaryk University, Brno

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Outline

1. Motivation
2. The Poincaré lemma
3. Covariant exterior derivative
   - Homogenous parallel transport equation
   - Inhomogenous parallel transport equation
4. Other applications
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Motivation

ODEs

Solution of

$$\frac{dx(t)}{dt} = f(t) \tag{1}$$

is

$$x(t) = C + \int_0^t dt' f(t'), \quad C = x(0) \in ker\left(\frac{d}{dx}\right). \tag{2}$$

ODEs 2

Consider equation

$$\frac{dx(t)}{dt} = f(t)x(t), \;\; x(0) = C. \tag{3}$$

We switch to integral equation

$$x(t) = C + \int_0^t x(t')f(t')dt', \tag{4}$$

and iterate

$$x(t) = C + \int fC + \int f \int fC + \dots . \tag{5}$$

Can we have such simple way of solving exterior differential equation by an 'integral' of some kind?

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The Poincaré lemma

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$$H^n(\mathbb{R}^k) = H^n(point) = \begin{cases} \mathbb{R}, & (n = 0) \\ 0 & (n > 0) \end{cases} \qquad (6)$$

$dim(H^k)$= no. of closed k-forms ($d\omega = 0$) that are not exact (not of the form $\omega = d\alpha$).

Not quite useful in computations.

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Homotopy Invariance Formula (for linear homotopy)

$$Hd + dH = I^* - s^*_{x_0}, \tag{7}$$

where

$$H\omega := \int_0^1 \mathcal{K} \lrcorner \omega|_{F(t,x)} t^{k-1} dt, \quad H : \Lambda^*(U) \to \Lambda^{*-1}(U), \tag{8}$$

for $\omega \in \Lambda^k(U)$, $\mathcal{K} := (x - x_0)^i \partial_i$, $k = deg(\omega)$, U - star-shaped, and linear homotopy $F(t,x) = x_0 + t(x - x_0)$ interpolates between Id and the constant map $s_{x_0} : x \to x_0$.

Why H is so interesting?

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$$H^2 = 0 \tag{9}$$

That gives $(Hd + dH = I - s^*_{x_0})$

$$(Hd)^2 = Hd, \quad (dH)^2 = dH. \tag{10}$$

Geometric decomposition

We have projectors Hd and dH onto

- Exact/closed vector space $\mathcal{E}(U) = im(dH) = ker(d)$,
- Antiexact module $\mathcal{A}(U) = im(Hd) = ker(H)$,
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Solve (on starshaped U):

$$d\alpha = J \tag{11}$$

- Conclusion: $dJ = 0$, i.e., $J \in \mathcal{E}(U)$, i.e., $J = dHJ$,
- $d(\alpha - HJ) = 0$, i.e., $\alpha - HJ \in \mathcal{E}(U)$,
- $\alpha = c + HJ$, where $c \in \mathcal{E}(U) = ker(d)$.

Replacing:

$$\begin{aligned} d &\longleftrightarrow \tfrac{d}{dx}, \\ \textstyle\int &\longleftrightarrow H, \end{aligned} \tag{12}$$

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$$d\alpha = J \tag{13}$$

What if $dJ \neq 0$?
Then

$$J = J_e + J_a \in \mathcal{E} \oplus \mathcal{A}, \quad J_a = HdJ \neq 0. \tag{14}$$

so

$$\underbrace{d\alpha - J_e}_{\mathcal{E}} - \underbrace{J_a}_{\mathcal{A}} = 0, \tag{15}$$

and therefore, $J_a = 0$ - a contradiction!
We must add additional (antiexact) term. There is plenty of options, however, one is

$$d\alpha + A \wedge \alpha = J_e + J_a, \tag{16}$$

for some $A \in \Lambda^1(U)$. ('Cartan-like minimal coupling') Situation similar to non-autonomous ODEs - time dependence is related to the interaction with other part of the system.

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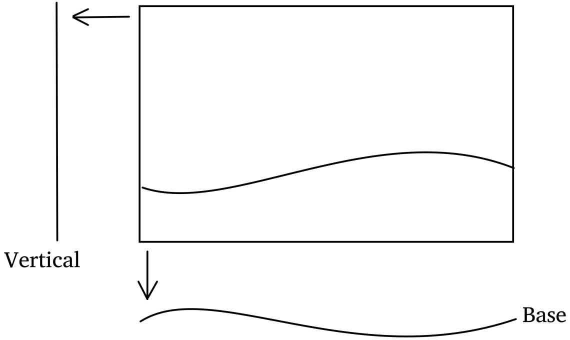
Covariant exterior derivative

Fibered set



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We are interested in V-valued differential forms: $\Lambda(U, V)$.

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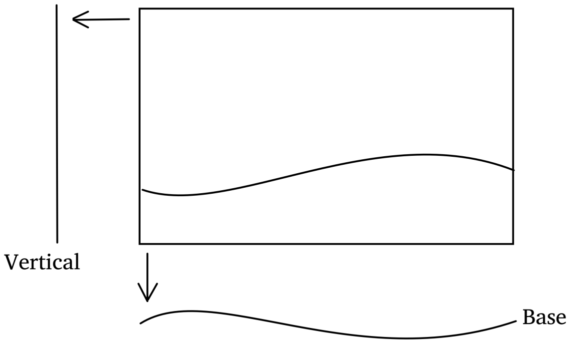

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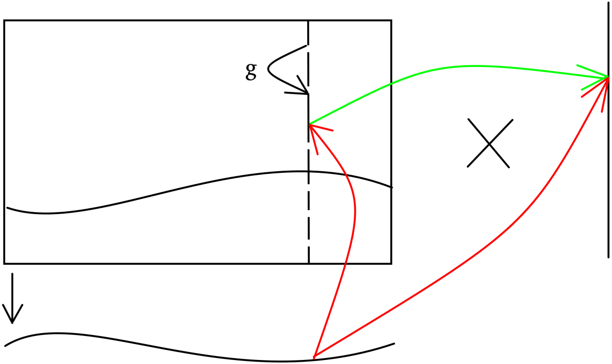
Vector-valued differential forms



Sections of associated vector bundle are in 1:1 correspondence with equivariant horizontal forms.

Covariant exterior derivative

$$d^{\nabla} := d + A \wedge \_, \tag{17}$$

for $A \in \Lambda^1(U, End(V))$ (usually with additional properties related to underlying bundle).

Reminder:

$$\begin{array}{ccc} \Lambda^0_b(P,V) & \xrightarrow{d^{\nabla}} & \Lambda^0_b(P,V) \\ \downarrow \Psi & & \Psi \downarrow \\ \Gamma(E) & \xrightarrow[\nabla]{} & \Gamma(E) \end{array}$$

- $G \to P \to M$ - a principal bundle
- $E = P \times_G V$ - associated vector bundle
- $\Lambda^0_b(P,V)$ - basic forms

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(Homogenous) parallel transport equation

$$d^{\nabla}\phi = 0, \quad \phi \in \Lambda^k(U, V), \tag{18}$$

with the "intial/boundary" condition $dH\phi = c \in \mathcal{E}(U, V)$.

Decompose

$$\phi = \phi_1 + \phi_2,$$

where

$$A \wedge \phi_2 = 0,$$

$$d\phi_1 + A \wedge \phi_1 = -d\phi_2.$$

An element

$$\phi_2 \in \mathcal{E}(U, V) \cap ker(A \wedge \_)$$

is a 'gauge mode' - nonuniquness of the solution.

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$$d^{\nabla}\phi = 0, \quad \phi \in \Lambda^k(U,V) \setminus ker(A \wedge \_), \tag{19}$$

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Replace with:

$$d\phi + \lambda A \wedge \phi = 0, \quad \lambda \neq 0. \tag{20}$$

Substitute a formal power series:

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \dots, \tag{21}$$

Compare the $O(\lambda^k)$ terms:

- $O(1)$: $d\phi_0 = 0$
- $O(\lambda^1)$: $d\phi_1 + A \wedge \phi_0 = 0$
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Substitute a formal power series:

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \dots, \tag{21}$$

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(Homogenous) parallel transport equation

$$d^{\nabla}\phi = 0, \quad \phi \in \Lambda^k(U,V) \setminus ker(A \wedge \_), \tag{19}$$

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$$\phi_1 = d\alpha_1 - H(A \wedge \phi_0), \tag{22}$$

  for $\alpha_1 \in \Lambda^{k-1}(U, V)$.
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- $O(\lambda^l)$: $\phi_l = d\alpha_l - H(A \wedge \phi_{l-1})$.

Solution

We have a power series solution:

$$\phi = \sum_{l=0}^{\infty} (-1)^l (H(A \wedge \_))^l c, \tag{23}$$

where $c = \sum_l d\alpha_l \in \Lambda^{k-1}(U, V)$ is an arbitrary form, and $(H(A \wedge \_))^0 = Id$.

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Exact-inhomogenous parallel transport equation

The unique solution of

$$d^{\nabla}\phi = J, \tag{24}$$

for $\phi \in \Lambda^k(U,V) \setminus ker(A \wedge \_)$, $A \in \Lambda^1(U, End(V))$, $J \in \mathcal{E}^{k+1}(U,V)$, with $dH\phi = c \in \mathcal{E}(U,V)$ is

$$\phi = \phi_H + \phi_I, \quad \phi_I = \sum_{l=0}^{\infty}(-1)^l (H(A \wedge \_))^l HJ, \tag{25}$$

where ϕ_H is a solution of homogenous equation $(J = 0)$.
The series in (25) is convergent for $||x - x_0|| < \frac{k}{||A||_\infty}$, where the supremum norm is taken over the line $L = \{x_0 + t(x - x_0)|t \in [0;1]\}$.

Inhomogenous parallel transport equation

The solution of the inhomogeneous covariant constancy equation

$$d^{\nabla}\phi = J, \quad d^{\nabla} = d + A \wedge \_, \tag{26}$$

where $\phi \in \Lambda^k(U,V)$, $A \in \Lambda^1(U,V)$, $J \in \Lambda^{k+1}(U,V)$ is given by

$$\phi = \phi_1 + \phi_2 + \phi_3, \tag{27}$$

where ϕ_1 fulfils

$$d^{\nabla}\phi_1 = J_e - d(\phi_2 + \phi_3), \tag{28}$$

and ϕ_2 fulfils (constraint)

$$A \wedge \phi_2 = J_a, \tag{29}$$

where $J_e := dHJ$ is the exact part of J, and $J_a := HdJ$ is the antiexact part of J. The $\phi_3 \in \ker(A \wedge \_)$ is an arbitrary form. Moreover $A \wedge \phi_1 \in \mathcal{E}^{k+1}(U,V)$ and $A \wedge \phi_2 \in \mathcal{A}^{k+1}(U,V)$.

Associated vector bundles

Is solution a base form on associated vector bundle?

- Equivariance of solutions: Results from the uniqueness of the solution (of parallel transport PDE).
- Horizontality: Not always! Not every solution corresponds to section of associated vector bundle - we get no miracles!

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Relation to Cartan structure equations

The torsion equation:

$$d\omega^{\mu} + \omega^{\mu}_{\nu} \wedge \omega^{\nu} = T^{\mu}, \tag{30}$$

where ω^{μ}_{ν} is fixed - second structure equation. It is solved by above methods.

The second structure equation can also be solved by homotopy operator, see our draft.

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Other applications

Other applications

Plenty of **equations of mathematical physics** on Riemannian manifolds, e.g., Maxwell equations:

$$DF = J, \tag{31}$$

where the Dirac(-Kähler) operator is

$$D = d - \delta. \tag{32}$$

Other applications

For Riemannian manifolds, **Hodge star duals** to parallel transport equation:

$$\delta\phi + i_{A^\sharp}\phi = J. \tag{33}$$

Use $i_{A^\sharp} \star \phi = \star(\phi \wedge A)$.

There is linear (co)homotopy operator for codifferential

$$h = \eta \star^{-1} H \star, \tag{34}$$

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Curvature (algebraic) equation treated as a differential equation:

$$F \wedge \phi = d^{\nabla} d^{\nabla} \phi = J. \quad (35)$$

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$$\begin{cases} d^{\nabla} \phi_2 = J \\ d^{\nabla} \phi_1 = \phi_2. \end{cases} \quad (36)$$

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More generally, having,

$$D_i = d + A_i \wedge \_, \quad \reflectbox{D}_i = \delta + i_{A^\sharp}\_ \tag{37}$$

we can construct **geometr-based differential equation**

$$D_1^{i_1} \reflectbox{D}_2^{i_2} \ldots \phi = J. \tag{38}$$

Again, replace equation with system of equations and apply previous ideas. Example:

$$d\delta\phi = J, \tag{39}$$

is replaced by

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- It is known that vertical derivative of jet bundle has a vertical homotopy operator - Vainberg-Tonti Lagrangian.
- It was however not known that the obstacle to variationality of a differential equation('as it stands') is associated with antiexact vertical form.
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Feynman graphs:

- Using integration by parts, we can express any Feynman integral I as a linear combination of master integrals $\vec{I} = \{I_1, \ldots, I_n\}$.
- The master integrals fulfil the equation

$$(d + A)\vec{I} = 0, \;\; A \in \Lambda^1(\mathbb{R}^N, \mathbb{R}^{n\times n}), \tag{41}$$

where N - number of kinematic variables. A is a flat connection, i.e., $dA + A \wedge A = 0$.

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- The Poincare lemma and resulting homotopy operators have nice 'operator' properties.
- On a star-shaped subset of a fibered set we can solve "geometry-based differential equation" using homotopy operator in the same way as integral is used for ODE.
- (Almost) any such local problem can be easily solved using our approach.
- "Sheafication" is still missing to go from local to global view. Then the topology of underlying space starts to play a role.
- What is not yet done: Einstein equations, Green's function for Laplace-Beltrami operator.

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Acknowledgement

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Bibliography

D.G.B. Edelen, *Applied Exterior Calculus*, Dover 2011

R.A. Kycia, *The Poincare lemma, antiexact forms, and fermionic quantum harmonic oscillator*, Results Math 75, 122 (2020)

R.A. Kycia, *The Poincare lemma for codifferential, anticoexact forms, and applications to physics*, Results Math. (2022)

R.A. Kycia, J. Silhan, *Inverting covariant exterior derivative*, arXiv: 2210.03663 [math.DG]

Thank You for Your Attention

Köszönöm a figyelmet

Backup

Link with operator calculus

Link with operator calculus

Mimic

$$\frac{d}{dx}\int_{x_0}^{x} f(t)dt = f(x), \quad \int_{x_0}^{x} \frac{df}{dt}(t)dt = f(x) - f(x_0), \tag{42}$$

for $f \in C^\infty(\mathbb{R})$.

Bittner's operator calculus

For linear spaces L_0 and L_1 we define linear operators

- $S : L_0 \to L_1$ - abstract derivative;
- $T_q : L_1 \to L_0$ - abstract integral parametrized by $q \in ker(S) \subset L_0$;
- $s : L_0 \to ker(S) \subset L_0$ - projection/limit condition;

that fulfills

$$ST = I, \quad TS = I - s. \tag{43}$$

Elements of $ker(S)$ are called constants (of S).

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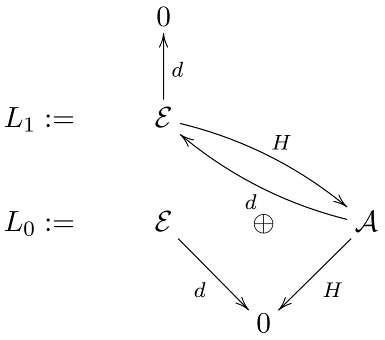
Define $L_0 = \mathcal{E} \oplus \mathcal{A}$ and $L_1 = \mathcal{E}$.



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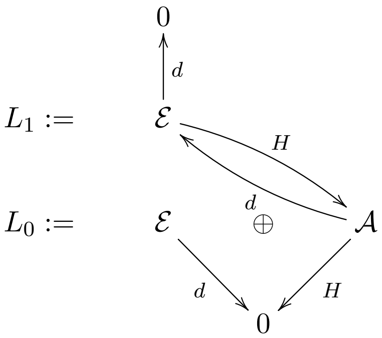

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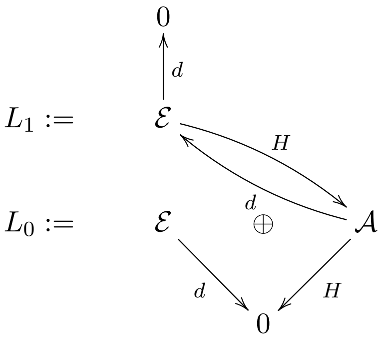

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In order to identify s operator, we use homotopy invariance formula as

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i.e.

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Chen's iterated integrals

Chen's iterated integrals (1977)

- For a (super)vector bundle $\pi : V \to M$, the connection one-form $\omega \in \Lambda^1(M, End(V))$, construct Path Space.
- Parallel transport operator $\Phi(t) : V_{\gamma(t=0)} \to V_{\gamma(t)}$, for a path $\gamma : [0;1] \to M$ fulfils:

$$\frac{d\Phi^\omega(t)}{dt} = i_t^* \partial_t \lrcorner \omega \wedge \Phi^\omega(t), \quad \Phi^\omega(0) = Id_V, \tag{46}$$

where $i_t : M \to M \times [0,1]$ is the inclusion $i_t(x) = (x,t)$, $\omega \in \Lambda^*(M \times [0,1], End(V))$.

- Then the solution is the operator series

$$\Phi^\omega(t) = \sum_{n=0}^{\infty} \Phi_n^\omega(t), \quad \Phi^\omega(0) = Id_V, \tag{47}$$

where $(t \geq s_1 \geq \ldots \geq s_n \geq 0)$:

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