Statistical Mechanics of the Universe[1]

April 2024

[1] Jürg Fröhlich, formerly at ETH Zurich

Credits and Contents

Credits

Work with *A. Alekseev, W. Aschbacher, O. Boyarsky, R. Brandenberger, A. Brandenburg, V. Cheianov, E. Lenzmann, B. Pedrini, I. Rogachevskii, O. Ruchayskiy, J. Schober, I. M. Sigal, T.-P. Tsai, Ph. Werner, H.-T. Yau*, and several other people, on and off, starting in 1997.
Useful discussions with *R. Durrer* and *N. Straumann*, over many years.

Contents

1. A List of Puzzles in Cosmology
2. [Setting the Stage – The Geometry of the Universe
3. The State of the Universe Shortly After Inflation]
4. Generation of Primordial Magnetic Fields
5. A tantalizing model of an axion and a scalar field
6. Matter-Antimatter Asymmetry and Dark Energy

   [Appendix: Pitfalls of Fuzzy Dark Matter]
7. Conclusions?

Disclaimer

This is an informal talk on matters I don't know very much about!

My interests in various *special problems of cosmology* arose accidentally: Since 1989/90 I was working on the theory of the quantum Hall effect (QHE). In 1998, I got interested in the seemingly purely academic question whether there are higher-dimensional cousins of the QHE $\rightarrow$ work with Alekseev and Cheianov. Ruth Durrer drew my attention to possible applications of our results to the problem of the origin of *cosmic magnetic fields*. In 1998, I also became interested in the *Mean Field Limit* of bosonic many-body systems, originally studied by Klaus Hepp (1974), and in solitary wave solutions of the limiting (Mean-Field) non-linear dynamics $\rightarrow$ *gravitational instabilities of boson- and neutron stars; possible models of axionic Dark Matter*.
In 2015, I got interested in the problem of *Dark Energy* (and possible interplays of DE and DM) and in the *Matter-Antimatter Asymmetry* in the Universe.

I have various proposals about how to attack these problems; and I hope they are reasonably realistic.

1. A List of Puzzles in Cosmology

$\geq$ 7 basic puzzles in cosmology:

1. Formation of "classical" structure from initial quantum state of Universe: What are *"(Cosmological) Events"* in QM?
   ($\nearrow$ "ETH - Approach to QM"!)
   What is *"Dark Matter"*, how does it produce "classical" structure?
2. Role of *Inflation* – what does it explain, is it natural, is it real?
3. Why is expansion of Universe *accelerated* – what is *"Dark Energy"*?
4. Origin of *matter-antimatter asymmetry* in the Universe?
5. Why are there comparable amounts of *Visible Matter*, *DM* and *DE* in the Universe? Was this and will this always be the case?
6. * Origin of *cosmic magnetic fields* ext. over intergalactic distances?
7. Cosmological hints at *Physics beyond the Standard Model* ?
   Neutrino masses, new degrees of freedom, such as *WIMP's*, *axions*, *new scalar fields*, *new gauge fields*, etc. ?

Comments on Puzzles 1 through 7

1. "ETH - Approach to QM" is expected to solve this problem! – Emergence of classical behaviour in Mean-Field Limit (i.e., for tiny density, tiny gravitational coupling constant) of models of Visible Matter and (Fuzzy) Dark Matter; (Sect. 6).
2. Standard wisdom on inflation: would explain homogeneity, isotropy and spatial flatness ($\Omega_0 = 1$) of the Universe. *Observational indications of inflation:* CMB, nearly scale-inv. (red-shifted) fluctuations, acoustic peaks.
3. Proposal of a model of dynamical Dark Energy in Sect. 5.
4. New scalar field as a chemical potential for matter-antimatter asymmetry and a candidate of Dark Energy (?); Sect. 5.
5. For me, this remains mysterious – vague ideas concerning "tracking DE"; see Sect. 5.
6. A QED axion as a source of cosmic magnetic fields; Sect. 4.
7. Clearly related to items 2 through 6! – Extra dimension(s)?

2. Setting the Stage – The Geometry of the Universe

From CMB: Up to an age of some $100'000$ years (before large-scale structures formed) Universe was remarkably *homogenous* and *isotropic*. Possible explanation: *Inflation!* Throughout, I will treat it as homogeneous and isotropic on very large distance scales; (e.g., dists. $\geq 10^7$ pc $\ll$ optical radius of Universe $< 10^{10}$ pc).
Consequence: Universe foliated in *space-like hypersurfaces*, $\{\Sigma_t\}_{t\in\mathbb{R}}$, orthogonal to a *time-like geodesic velocity field* U, on which induced metrics are all proportional to one another $\Rightarrow$

$$d\tau^2 = dt^2 - a^2(t)ds^2, \tag{1}$$

where t is cosmological time, $a(t)$ is a scale factor, and ds^2 is the metric of 3D Riemannian manifold, Σ, of *constant curvature*,

$$k = \frac{\varepsilon}{R^2}, \qquad \varepsilon = 0, \pm 1.$$

Geometry of Universe – ctd.

Meaning of parameter ε:
$\varepsilon = -1$: open, expanding for ever/ $\varepsilon = 0$: flat, expanding /
$\varepsilon = 1$: closed, evt. collapsing; $R =$ "curvature radius" of Σ.

Plug ansatz (1) into *Einstein's Field Eqs.*, with energy-momentum tensor, $T = (T^{\mu}_{\nu})$, given by

$$T = \mathrm{Diag}(u, -p, -p, -p),$$

and appropriate *equations of state* relating ρ to p.

$\Rightarrow \sim$ *Friedmann Eqs.:*

$$\boxed{3H^2 + 3\frac{k}{a^2} = \kappa u + \Lambda,} \tag{2}$$

where $\kappa = 8\pi G_{Newton}$, $H(t) := \frac{\dot{a}(t)}{a(t)}$: Hubble "constant",
Λ: cosmological constant;

Geometry of Universe – ctd.

and

$$\boxed{2\dot{H} - 2\frac{k}{a^2} = -\kappa(u + p)} \tag{3}$$

Inflation $\Rightarrow k = 0$; we also set $\Lambda = 0$. By Eq. (2),

$$u_{\text{crit.}} = \frac{3}{\kappa} H^2, \qquad \text{corresp. to } k = 0, \Lambda = 0.$$

Density parameter

$$\Omega_0 := \frac{u}{u_{\text{crit.}}}.$$

From data: $\Omega_0 \approx 1$, as would be explained by Inflation! This implies that, besides Visible Matter (VM, $\approx 5\%$), Dark Matter (DM, $\approx 27\%$), there must also exist Dark Energy (DE, $\Rightarrow \approx 68\%$), as confirmed by data from type IA supernovae (Perlmutter, Schmidt, Riess), CMB and Baryon oscillations (BAO – oscillations in power spectrum of matter).

Equations of State

(i) VM and DM: $p \approx 0$

(ii) Radiation: $T^{\mu}_{\mu} = 0 \quad \Rightarrow \quad p = \frac{u}{3}$ (conformal invariance)

(iii) DE (mimics Λ): $p \approx -u$

DE apparently dominates ($\approx 68\%$) $\Rightarrow$ Must solve *Friedmann Eqs.* with

$u + p = \delta u,\ 0 < \delta < 4/3$, (at present $\delta \approx 1/3$), yielding

$$\begin{aligned} a(t) &= a(t_0)\,(t+\tau/t_0+\tau)^{2/3\delta}, \quad \text{for some const. } \tau\,, \\ H(t) &= (2/3\delta)(t+\tau)^{-1}, \\ u(t) &= (4/3\kappa\delta)(t+\tau)^{-2} = \text{const.}\, a(t)^{-3\delta}\,. \end{aligned} \tag{4}$$

For Rad.: $\delta = \frac{4}{3}$, $u(t) \propto a(t)^{-4} \propto t^{-2}$ (redshift!); for VM & DM: $\delta = 1$, $u(t) \propto a(t)^{-3} \propto t^{-2}$; for DE only: $\delta = 0, u(t) = const., H = const.$

Assuming Universe is in thermal equilibrium in radiation-dominated phase, before recombination, *Stefan-Boltzmann* implies that

$$\boxed{T(t) \propto u^{1/4} \propto \frac{1}{\sqrt{t}}\,, \quad (\text{but } T(t) = \text{const., for DE, i.e., } \delta = 0!)} \tag{5}$$

3. The State of the Universe Shortly After Inflation

U: time-like geodesic velocity field; Σ_t: space-like surface ($\nearrow$Eq. (1)); $dvol_t(x)$ = volume form of the metric $a(t)^2 ds^2$ on Σ_t. – Initially, all quantities encountered below are to be understood as *qm operators*..

Quantum State of early Universe in radiation-dom. phase (before matter decouples): Local thermal equilibrium (LTE) at a temperature $T \approx$ (5). In order to identify this state, must know which quantities are (approx.) conserved in the hot, early Universe. Let's imagine these quantities correspond to approximately conserved currents, $J^\mu_a, a = 1, 2, \ldots$,

$j_a :=$ 3-form dual to J_a ; q.m. energy density: $T_{00} \propto T(U, U)$.

Then LTE at time t is described by the (ill-def.) *"density matrix"*

$$P_{LTE} \propto exp\left(- \int_{\Sigma_t} \beta(x) \left[T_{00}(x)\, dvol_t(x) - \sum_a \mu_a(x)\, j_a(x) \right] \right), \qquad (6)$$

where $\beta(x)$ is a (space-) time-dep. inverse temperature; "fields" $\mu_a(x)$ are local (space-time dep.) chemical potentials conjugate to (approx.) conserved currents J_a; normalisation factor multiplying R.S. of (6), chosen such that trace of P_{LTE} is $= 1$, is called inverse partition function.

Conserved and Anomalous Currents

From now on, adopt *thermodynamical interpretation* of $\beta(x), \mu_a(x)$ as state parameters/ "moduli".

A current, J, is said to be *conserved* iff

$$\nabla_\mu J^{\mu\cdots} = 0 \quad \Leftrightarrow \quad \mathrm{d}j^{\cdots} = 0 \quad \text{(d: exterior derivative)}$$

Conserved vector currents:
(i) Electric current (density) J;
(ii) $J_B - J_L \leftrightarrow$ matter-antimatter asymmetry;...

Digression on axial currents: In the presence of gauge fields and for massive matter fields, *axial currents*, J_5^μ, tend to be *anomalous*:

$$\nabla_\mu J_5^\mu = \frac{\alpha}{4\pi}\varepsilon^{\mu\nu\sigma\rho}\mathrm{tr}(F_{\mu\nu}F_{\sigma\rho}) + \text{terms} \propto \text{masses}, \qquad (7)$$

(chiral anomaly). Here F is the field tensor of a gauge field, A, and α is (an analogue of) the fine structure constant.

An example of an anomalous current is the leptonic axial current, $J_{L,5}^\mu$, sensitive to the asymmetry between left-chiral and right-chiral leptons.

Conservation Laws Assoc. With Anomalous Currents

Let j_5 be the 3-form dual to an anomalous current J_5^μ. Let $\mathrm{tr}(A \wedge F)$ be the Chern-Simons 3-form of an abelian gauge field A; (with components $\mathrm{tr}(A_{[\mu} F_{\nu\rho]})$). If *masses of matter fields are negligible* then Eq. (7) $\Rightarrow$

$$\mathrm{d}\Big(j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)\Big) = 0,$$

i.e., the axial current dual to $j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)$ is *conserved*, though *not* gauge-invariant. However,

$$Q_5 := \int_{\Sigma_t} \Big(j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)\Big) \qquad (8)$$

is a gauge-invariant, conserved charge.
In order for the state (6) to be gauge-invariant, we must require that

$$\mathrm{d}(\beta\mu_5) \wedge F|_{\Sigma_t} = 0, \qquad (e.g., \ \beta\mu_5 \text{ only dep. on time }, t),$$

where μ_5 is the chemical potential conjugate to $j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)$.

Conserved Currents & Conjugate Chemical Potentials

Remark: For abelian gauge fields, e.g., e.m. vector potential, the chiral anomaly can also be expressed in terms of anomalous commutators:

$$[J_5^0(\vec{y}, t), J^0(\vec{x}, t)] = \frac{\alpha}{\pi} \vec{B}(\vec{y}, t) \cdot \vec{\nabla} \delta(\vec{y} - \vec{x}) \,. \qquad (9)$$

End of **digression.**

Henceforth, gauge fields and currents will be treated as classical, (i.e., as expectations of qm operators in the state of the Universe). – Quantum cosmology remains to be developed!

Examples of conserved currents and conjugate chemical potentials:

I. Electric vector current density:

$J^\nu \; \leftrightarrow \; \mu_{el} = 0$ (local electric neutrality!)

II. $J_B^\nu - J_L^\nu \; \leftrightarrow \; \mu_{B-L}$ (tunes matter-antimatter asymmetry; μ_{B-L} related to a scalar field, σ, connected to DE (?))

III. Leptonic axial current density, $J_{L,5}^\nu$, dual to $j_{L,5} - \frac{\alpha}{2\pi} A \wedge F$, where A is the electromagnetic vector potential, masses *neglected*:

$J_{L,5}^\nu \; \leftrightarrow \; \mu_5$ (tunes left-right asym., $\mu_5 \propto \dot{\theta}$, θ an "axion" field) → (23)

4. Generation of Primordial Magnetic Fields from Axions

Maxwell Equations in an Expanding Universe, with $\Sigma \simeq \mathbb{E}^3$ flat:

$$\vec{\nabla} \wedge \vec{E} + \dot{\vec{B}} + \frac{3}{2} H \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \tag{10}$$

$$\vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} - \frac{3}{2} H \vec{E} + M^2 \vec{A}^{\perp} = \vec{J}, \quad \vec{\nabla} \cdot \vec{E} = \rho. \tag{11}$$

Here H is the Hubble "constant", $\vec{A}^{\perp}$ is the electromagnetic vector potential in the Coulomb gauge, $\vec{J}$ is the electric current density, ρ is the charge density, and $M \geq 0$ is a photon mass. It is reasonable to assume that in a hot plasma $\rho \equiv 0$. We have to find an expression for the current density $\vec{J}$. There is an Ohmic contribution to $\vec{J}$, but also one that mirrors a possible left-right asymmetry (chiral magnetic effect). For simplicity, we assume that the primordial plasma is $\sim$ at rest in the coordinates introduced in (1), above. Then Eqs. (6) (P_{LTE}) and (9) (anomalous comm.) imply ($\nearrow$ Vilenkin (1980), ACF)

$$\boxed{\vec{J} = \sigma \vec{E} + \frac{\alpha}{\pi} \mu_5 \vec{B}} \tag{12}$$

Spatial dependence of μ_5 neglected, but depends on t! *Is it a "state modulus"?*

An Instability

Plugging (12) into (11), taking the curl of the first eq. in (11), and using the first eq. in (10), one finds:

$$-\Delta\vec{B} + \ddot{\vec{B}} + (2h+\sigma)\dot{\vec{B}} + \dot{h}\vec{B} + h(h+\sigma)\vec{B} - \mu_5\vec{\nabla}\wedge\vec{B} + M^2\vec{B} = 0, \quad (13)$$

with $h := \frac{3}{2}H$. We solve (13) by Fourier transformation[2]:

$$\vec{B} = \vec{b}\,e^{i(kz-\omega t)}, \text{ where } \vec{b}\perp\vec{e}_3 \quad (\vec{e}_3 = z- \text{ axis}) ,$$

using that $\vec{\nabla}\cdot\vec{B} = 0$. We then find that

$$\omega(k) = -i\left(h+\frac{\sigma}{2}\right) \pm \sqrt{-\left(h+\frac{\sigma}{2}\right)^2 + k^2 + M^2 + h(h+\sigma) + \dot{h} \pm \mu_5|k|} \quad (14)$$

We observe that the expansion of the Universe (i.e., $H > 0$) and Ohmic conductivity of the primordial plasma lead to power-law (actually, exp. if $h =$const.) damping of $\vec{B}$ in time, provided

$$\mu_5|k| < k^2 + M^2 + h(h+\sigma) + \dot{h} \quad (15)$$

[2]Time-dependence of μ_5 and of $h, \dot{h}$ assumed to be negligible

Instability – ctd

We will see that it is likely that $\mu_5 \searrow 0$, as $t \nearrow \infty$. Hence evolution of electromagnetic field is damped for large times if the mass M of the photon were positive. –*However*, for $M = 0$, one encounters a (transitory exp.) *instability* in the solutions of Eq. (13) for wave vectors k satisfying

$$\boxed{\frac{\mu_5 - \sqrt{\mu_5^2 - K}}{2} < |k| < \frac{\mu_5 + \sqrt{\mu_5^2 - K}}{2},} \qquad (16)$$

where $K := 4\big[h(h+\sigma) + \dot{h}\big]$; growth rate $\propto \sqrt{\mu_5/(h + \frac{\sigma}{2})}$.
This is a *mechanism for the growth of very homogeneous primordial magnetic fields* from quantum fluctuations, (which, at late times, exhibit power-law decay dictated by H).

A systematic study of *Relativistic Magneto-Hydrodynamics* in the presence of the chiral magnetic effect (terms in the equations of motion proportional to μ_5 $\propto$ time-derivative of axion field!) has been carried out with and by Boyarsky, Brandenburg, Ruchayskiy, Schober, and others.
We have identified various novel dynamos driven by chiral asymmetry.

Possible Origins of μ_5

Next, we search for possible origins of a non-trivial "chemical potential" μ_5. Let us look for a generally covariant form of Eq. (12), i.e., of $\vec{J} = \sigma\vec{E} + \frac{\alpha}{\pi}\mu_5\vec{B}$, linear in the field tensor $F_{\nu\lambda}$:

$$J^\nu(x) = \frac{\alpha}{2\pi}\varepsilon^{\nu\lambda\rho\tau}(\partial_\lambda\theta)(x)F_{\rho\tau}(x) - \sigma F^{\nu\lambda}(x)V_\lambda(x) + \rho(x)V^\nu(x). \tag{12'}$$

Here V^ν is the four-velocity field and ρ the charge density of the primordial plasma; θ is a pseudo-scalar (axion) field. Imposing local electric neutrality, i.e., $\rho \equiv 0$, we find that

$$\begin{aligned} \vec{J} &= \sigma\big(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\big) + \frac{D\theta}{Dt}\vec{B} \\ &+ \vec{\nabla}\theta \times \big(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\big), \end{aligned}$$

where

$$\frac{D\theta}{Dt} \equiv \mu_5 := \frac{\partial\theta}{\partial t} + \vec{V}\cdot\vec{\nabla}\theta. \tag{17}$$

Possible Origins of μ_5 – ctd.

We consider the special situation where $\sigma = \rho \equiv 0$. Then the Maxwell equations, with J^ν as in (12'), can be derived by varying the following action functional w. r. to the em vector potential A:

$$S(A,\theta) := \int \left[F_{\nu\lambda}(x)\, F^{\nu\lambda}(x)\, a(t)^3 d^4x - \frac{\alpha}{2\pi}\theta(x) F(x) \wedge F(x) \right], \qquad (18)$$

with $F \wedge F$ dual to $4\vec{E} \cdot \vec{B}$. Next, we search for an eq. of motion for θ. Let

$$q_5(t;\mu_5) := \text{ spatial average of } \rho_5(\vec{x},t) = \overline{\langle J_5^0(\vec{x},t)\rangle}_{\beta,\mu_5}$$

In the radiation phase ($a(t) \propto \sqrt{t}$, $T(t) \propto 1/\sqrt{t}$)

$$q_5(t;\mu_5) \approx \mu_5 \frac{\partial q_5}{\partial \mu_5}(t;\mu_5 = 0) \approx const.\, \mu_5\, T^2,$$

with $q_5(t;\mu_5 = 0) = 0$. By (17) (for an incompressible plasma),

$$\mu_5 = \overline{\dot{\theta}}.$$

Thus, neglecting terms $\propto T \cdot \dot{T}$, one derives from the chiral anomaly that

$$a(t)^{-2}\, \overline{\ddot{\theta}} = const.\, \dot{\mu}_5\, T^2 = const.\, \dot{q}_5 = const.\, \frac{\alpha}{\pi}\, \overline{\vec{E} \cdot \vec{B}}. \qquad (19)$$

Axion Equation of Motion

By (19), the field equation for θ in conformal time must look like

$$\Box\theta + \mathcal{U}'(\theta) = \frac{\alpha}{2\pi}\vec{E}\cdot\vec{B}, \tag{20}$$

a non-linear wave equation; *or like*

$$a(t)^{-2}\,\ddot{\theta} - D\Delta\dot{\theta} + \mathcal{U}'(\theta) = \frac{\alpha}{2\pi}\vec{E}\cdot\vec{B}, \tag{20'}$$

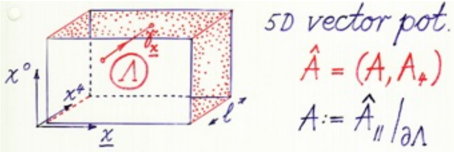
a non-linear diffusion eq., $D =$ diffusion constant, (with $\overline{\mathcal{U}'(\theta)} = 0$). ...
Eq. (20) is reminiscent of the field equation for an *axion* $\rightarrow$ identify θ with a *pseudo-scalar axion field.*
The Maxwell equations, with J^ν as in (12') ($\sigma \equiv 0, \rho \equiv 0$), and Eq. (20) can be derived by varying the action functional

$$S_{\text{tot}}(A,\theta) := S(A,\theta) + \int a(t)^3 d^4x \left[\partial_\nu\theta\partial^\nu\theta + \mathcal{U}(\theta)\right]. \tag{21}$$

Remark: For $\mathcal{U} \equiv 0$, this action can be derived from Maxwell theory in 5D with a 5D Chern-Simons term by dimensional reduction, with

$$\theta(x) := \overline{\hat{A}_4(x,\cdot)} \;\Rightarrow\; \mu_5 = \overline{\hat{E}_4(x,\cdot)} \;\rightarrow \text{5D QHE (bulk-edge duality)!}$$

A 5D Cousin of the QHE



5D bulk, Λ, filled with heavy 4-component Dirac fermions coupled to $\hat{A}$ $\Rightarrow$ *PT*- breaking! $\rightarrow$ 5D analogue of anomalous Hall (bulk) current:

$j = \sigma_H \hat{F} \wedge \hat{F}$, $\sigma_H \sim$ 5D "Hall conductivity" $\Rightarrow$ chiral surface currents.

Visible world located on $\partial\Lambda$, of 5D Univ.; light left-chiral- and right-chiral surface modes on different boundary branes; (masses gen. by tunneling)!

Instead of $-\frac{\alpha}{\pi}\,\theta\,\vec{E}\cdot\vec{B}$ in the action (18), we could add ($\nearrow$ chir. anomaly)

$$\int \partial_\nu \theta(x) J^\nu_{L,5}(x)\, a(t)^3 d^4x = \int \mathrm{d}\theta \wedge j_{L,5}\ , \tag{22}$$

which would introduce additional terms $\propto$ lepton masses. Formula (22) shows that $\dot{\theta}$ can be interpreted as a "chemical potential" for $j_{L,5}$.

5. A tantalizing model of an axion and a scalar field

See, e.g., *"After the Dark Ages"*, J. Phys. A, 2024, dedicated to Michael Berry.

Let's introduce a complex field

$$Z = e^{-(\sigma + i\theta)/f}, \qquad (5.1)$$

where the scalar field σ gives rise to *Dark Energy*, the axion field θ is assumed to account for *Dark Matter*; f is a constant with the dimension of an inverse length, rendering $(\sigma + i\theta)/f$ dimensionless. The model is only viable for values of f at least as large as the Planck mass, M_{Planck}. We set

$$\zeta_\mu := Z^{-1}\partial_\mu Z = -f^{-1}(\partial_\mu \sigma + i\partial_\mu \theta), \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \ \mu = 0, \ldots, 3, \ x^0 = t.$$

Let $U(r)$ be a non-negative polynomial in the variable r with the property that $U(r = 0) = 0$. Typical examples are

$$U(r) := \Lambda r, \qquad U(r) \equiv U^{(4)}(r) := \frac{\Lambda}{r_0^2 + \nu}\left[r^2(r - r_0)^2 + \nu\, r^2\right], \qquad (5.2)$$

where Λ is a constant with the dimension of a fourth power of mass, and $r_0 > 0$ and $\nu \geq 0$ are dimensionless constants.

Coupling to a gauge field

Let A be a gauge field (e.g., the weak $SU(2)$-gauge field), and let j be the 3-form dual to an anomalous current, J^μ, (e.g., the baryon current) with

$$dj = \frac{\alpha}{4\pi} \mathrm{Tr}\big(F_A \wedge F_A\big) + \mathcal{O}(M)\,, \tag{5.3}$$

where F_A = field strength of A, α = dimensionless coupling constant, M = a typical mass parameter of matter fields contributing to $\mathfrak{J}^\mu$. Written in components, Eq. (5.3) takes the form

$$\partial_\mu J^\mu = \frac{\alpha}{4\pi} (F_A)_{\nu\kappa} \widetilde{F}_A^{\nu\kappa} + \mathcal{O}(M)\,, \tag{5.4}$$

where $\widetilde{F}_A = \big(\widetilde{F}_A^{\nu\kappa}\big)$ is the dual field tensor. We introduce an *action functional*, S, for the field Z.

$$S(\overline{Z}, Z) := \int \Big[\frac{f^2}{2}\,\overline{\zeta}_\mu \cdot \zeta^\mu - U(|Z|) - \lambda(\partial_\mu \mathrm{Im} Z) \cdot \mathfrak{J}^\mu\Big] \sqrt{-g}\, d^4x\,, \tag{5.5}$$

where U is as in (5.2), and λ is a dimensionless coupling constant.

Action functional of the theory

When expressed in terms of σ and θ the action S takes the form

$$S(\sigma,\theta) = \int \Big[\frac{1}{2}(\partial_\mu\sigma\cdot\partial^\mu\sigma + \partial_\mu\theta\cdot\partial^\mu\theta) - U(e^{-\sigma/f}) + \\ + \lambda\partial_\mu\big(e^{-\sigma/f}\,\sin(\theta/f)\big)\cdot\mathfrak{J}^\mu\Big]\sqrt{-g}\,d^4x\,, \tag{5.6}$$

with $U(e^{-\sigma/f}) \simeq \Lambda e^{-2\sigma/f}$, as $\sigma \to +\infty$, for U as in (5.2). In the third term ($\propto \sin(\theta/f)$) we integrate by parts and then use Eq. (5.4) to obtain

$$\int \Big[\cdots - \frac{\lambda\cdot\alpha}{4\pi}e^{-\sigma/f}\sin(\theta/f)(F_A)_{\nu\kappa}\widetilde{F}_A^{\nu\kappa}\Big]\,,$$

up to terms of $\mathcal{O}(M)$. After a phase transition (e.g., electro-weak transition at $T_c \approx 160$ GeV), A acquires a mass. Integrating out A, taking into account all gauge field configurations with *non-vanishing instanton numbers,* but treating $g_{\mu\nu}$, σ and θ as (classical) background fields, we get an effective action of the form

The fields σ and θ as "gravitational degrees of freedom"?

$$S_{eff}(\sigma,\theta) = \int \Big[\frac{1}{2}(\partial_\mu\sigma\cdot\partial^\mu\sigma + \partial_\mu\theta\cdot\partial^\mu\theta) - \\ - U(e^{-\sigma/f}) - V(\sigma,\theta)\Big]\sqrt{-g}\, d^4x\,, \tag{5.7}$$

where $V(\sigma,\theta)\,(=\mathcal{O}(\theta^2),\ \text{for } \theta\approx 0)$ has a quadratic local minimum at $\theta = 0$ and is a periodic function of the axion field of the form

$$V(\sigma,\theta) \simeq \frac{1}{2}\mu^4 e^{-2\sigma/f}\sin^2(\theta/f)\,, \tag{5.8}$$

for small values of $e^{-\sigma/f}\,|\sin(\theta/f)|$, μ a const. with dimension of mass.

Remark: The functionals in (5.5), (5.6), (5.7) do *not* give rise to a renormalizable QFT. It is tempting to think of σ and θ as describing gravitational degrees of freedom in a theory with extra dimensions.
Question: Might string theory provide an ultraviolet completion for the effective theory of $(g_{\mu\nu},\sigma,\theta)$? (See B-F-H for some details.)
Here S_{eff} will only be used as action of a classical field theory governing the evolution of the cosmos. For simplicity, θ will be neglected.

6. Matter-Antimatter Asymmetry and Dark Energy

The current $J_{B-L} := J_B - J_L$ is conserved, and the corresponding charge, Q_{B-L}, is a measure of Matter-Antimatter Asymmetry. Assuming that this asymmetry originates in a phase when the Universe was in a state of local thermal equilibrium, i.e., before matter decoupled, it is natural to imagine that a chemical potential, μ_{B-L}, conjugate to Q_{B-L} tunes the Matter-Antimatter Asymmetry. Possible choices for μ_{B-L} might be

$$\mu_{B-L} = \dot{\sigma}, \text{ or } \mu_{B-L} = \sigma \cdot \dot{\phi}, \quad \text{where } \phi \text{ is another scalar field.} \tag{23}$$

In this section the field σ is argued to tune the Matter-Antimatter Asymmetry and that it also gives rise to roughly the right amount of *Dark Energy*. Setting $\theta \equiv 0$, its action functional is

$$S(\sigma; g) := \int \sqrt{-g}\, \mathrm{d}^4x\, \mathcal{L}(\sigma, \partial_\mu \sigma; g)(x), \quad \text{where} \tag{24}$$

$$\mathcal{L}(\sigma, \partial_\mu \sigma; g)(x) := \tfrac{1}{2} \partial_\mu \sigma(x) g^{\mu\nu}(x) \partial_\nu \sigma(x) - \Lambda e^{-(\sigma(x)/f)},$$

$f \approx M_{\text{Planck}}$ is a constant, and $(g_{\mu\nu})$ is the metric on space-time.

Equation of Motion in a Homogeneous, Isotropic Space-Time

We make the ansatz that σ only depends on cosmological time t and that the metric of space-time satisfies the Friedmann eqs. Then

$$S(\sigma, g) \equiv S(\sigma) = \mathrm{vol}(\Sigma) \int \mathrm{d}t \, \underbrace{a(t)^3 \{\frac{1}{2}\dot{\sigma}(t)^2 - \Lambda e^{-(\sigma(t)/f)}\}}_{\equiv L(\dot{\sigma}, \sigma, t)} \qquad (25)$$

The Euler-Lagrange equation of motion reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\sigma}} - \frac{\partial L}{\partial \sigma} = 0$$

$$\Leftrightarrow \quad \boxed{\ddot{\sigma}(t) + 3H(t)\dot{\sigma}(t) - \frac{\Lambda}{f} e^{-(\sigma(t)/f)} = 0,} \qquad (26)$$

with $3H(t) = \frac{2}{\delta} t^{-1}$; (w.l.o.g. $t + \tau \mapsto t$!) The parameter δ has been introduced in our discussion of equations of state:

$$u + p =: \delta u, \qquad \text{(with } \delta \approx \frac{1}{3}, \text{ at present)}$$

The Energy-Momentum Tensor of σ

Here u is the total energy density and p is the total pressure. These quantities are constrained by the Friedmann equations (2), (3). The energy-momentum tensor T is given by

$$T \equiv (T^{\mu}_{\ \nu}) = \mathrm{Diag}(u, -p, -p, -p).$$

The contribution of the field σ to T is calculated from the formula

$$T_{\mu\nu} = \frac{\delta S(\sigma, g)}{\delta g^{\mu\nu}} \;\Rightarrow\; T^{\mu}_{\ \nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \sigma(x))} \cdot (\partial_\nu \sigma)(x) - \delta^{\mu}_{\ \nu} \mathcal{L}(\sigma, \ldots)(x).$$

For our special ansatz, $\sigma = \sigma(t)$ (indep. of $\vec{x}$), this yields

$$u_\sigma = \frac{1}{2}\dot{\sigma}^2 + \Lambda e^{-(\sigma/f)}, \quad p_\sigma = \frac{1}{2}\dot{\sigma}^2 - \Lambda e^{-(\sigma/f)}. \tag{27}$$

Setting $u = u_\sigma + u_M$, $p = p_\sigma + p_M$, ($u_M :=$ energy density of matter, $p_M \approx 0$, Radiation neglected), the Friedmann equations yield

$$-\frac{2}{\kappa}\dot{H} = u + p = \dot{\sigma}^2 + u_M = \delta u = \frac{3\delta}{\kappa} H^2. \tag{28}$$

A Special Solution of the Equation of Motion (26)

[A special solution of Eq. (26) is given by

$$\sigma(t) \equiv \sigma^{(0)}(t) = \sigma_0\, \ell n(\frac{t}{t_0}),\ \text{with}\ \sigma_0 = 2f\,,\ t_0 = \sqrt{\frac{4-2\delta}{\delta\,\Lambda}}\, f \quad (29)$$

For this solution, we have that

$$u_\sigma(t) + p_\sigma(t) = \delta\, u_\sigma(t), \quad \forall\delta\,, \quad (30)$$

with

$$u_\sigma(t) = \frac{4}{\delta} f^2 t^{-2}, \quad p_\sigma(t) = (4 - \frac{4}{\delta}) f^2 t^{-2}.]$$

Thus, *the Friedmann equations are solved, provided*

$$\boxed{u_M, p_M = 0,\ \text{and}\ \ f^2 = (3\delta\kappa)^{-1}} \quad (31)$$

Tantalizingly, $f^2 \approx \kappa^{-1}$, for $\delta \approx \frac{1}{3}$ – as observed at present!

Interpretation of results (29), (30) and (31)

Remarks:

I. Relation (31) suggests that the field σ is a *gravitational degree of freedom*. In previous work with *Chamseddine* and *Grandjean*, it has been argued that $exp[-(\sigma/f)]$ is related to the scale of an extra dimension (chosen to be discrete in CFG), and that $f \approx \kappa^{-(1/2)}$ is a consequence of deriving the action functional for σ from a higher-dimensional *Einstein-Hilbert action* by "dimensional reduction". This fits well with the idea that the QED axion introduced in the Section 4 can be interpreted as arising from electromagnetism (with Chern-Simons term) on a 5D space-time, with a continuous or discrete extra dimension.

II. Results (30) and (31) suggest that, as time $t \to \infty$ (when matter and radiation become negligible), the solution $\sigma^{(0)}$ is an *"attractor"* in solution-space. This expectation is supported by the following result.

[Linear and Non-Linear Stability of $\sigma^{(0)}$

Theorem: General solutions, $\sigma(t)$, of (26) approach $\sigma^{(0)}(t)$, as $t \to \infty$.

Linear Stability:
Inserting the ansatz $\sigma(t) := \sigma^{(0)}(t) + \sigma^{(1)}(t)$, with $\sigma^{(1)}(t) \ll \sigma^{(0)}(t)$, for large t, into (26) and linearizing in $\sigma^{(1)}$, we find that

$$\sigma^{(1)}(t) \propto t^{-\alpha}, \quad \alpha = \beta \pm \sqrt{\beta^2 - 4\beta}, \quad \beta := \delta^{-1} - \frac{1}{2} > \frac{1}{4}.$$

Note that $\Re\alpha > 0$, $\forall \delta \leq \frac{4}{3}$, hence $\sigma^{(1)}(t) \searrow 0$, as $t \to \infty$.
If $\beta < 4$, i.e., $\delta > \frac{2}{9}$, then $\Im\alpha \neq 0 \Rightarrow \sigma^{(1)}$ describes *oscillations* (with a tiny time-dependent mass $\propto f(t_0/t)^2$) that die out like $t^{\frac{1}{2}-\delta^{-1}}$. These oscillations may contribute to Dark Matter.

Non-Linear Stability:

$$u_\sigma = \frac{1}{2}\dot{\sigma}^2 + \Lambda e^{-(\sigma/f)}$$

is a *Lyapunov functional* that decreases in time on solutions of (26).
All solutions of (26) are bounded above by $\ell n(\frac{t}{t_*})$, for some t_*.

Matter-Antimatter Asymmetry and Slow Roll of σ

To the action $S(\sigma, g)$ one can add the "topological term"

$$G \int \mathrm{d}\sigma \wedge j_{B-L}, \qquad \text{where } G \text{ is a constant.} \tag{32}$$

This term does not appear in the equation of motion for σ, because it is a *pure surface term*. However, it *will* appear in our formula for the state, P_{LTE}, of the Universe describing local thermal equilibrium. In the formula for P_{LTE}, the time derivative of σ plays the role of a *time-dependent chemical potential* conjugate to the conserved charge Q_{B-L}. Before matter decouples, $\dot{\sigma}$ could be large and, hence, might trigger a substantial asymmetry between Matter and Antimatter.

Yet, there is another problem we have to tackle! Since the Friedmann equations are automatically satisfied for the solution $\sigma^{(0)}$ of (26) displayed in (29), provided Relation (31) between f, δ and κ holds – *without* introducing additional fields – one may worry that there won't be room for *Radiation* and *Visible -*, as well as *Dark Matter* in the Universe.

Slow Roll –ctd.

However, one expects that, *for small* t, $H(t)$ deviates from $\frac{2}{\delta}$, and the true solution, $\sigma(t)$, of (26) deviates from $\sigma^{(0)}(t)$, so that ordinary *matter & radiation* may make a *non-vanishing contribution* to ρ and p!
If, for some reason, this did not work one would have to impose a "slow-roll" condition on $\sigma(t)$, effective at *early times*.
Recall that, among other desiderata, one wants to make sure that the very early Universe inflates. To guarantee this, one usually introduces an "inflaton" field.
Here we tentatively introduce an additional scalar field, ϕ, (possibly a "modulus" of the state of the early Universe), which conspires with the field σ to produce an additional term

$$G \int \sigma \cdot \mathrm{d}\phi \wedge j_{B-L}$$

in the action functional, where G is a constant. This term can trigger *Matter-Antimatter Asymmetry*, just like (32), with $\mu_{B-L} = G\,\sigma\dot{\phi}$.
In contrast to (32), it also modifies the Equation of Motion of σ! For a homogeneous, isotropic Universe, one finds:

Inflation, etc.

$$\ddot{\sigma}(t) + 3H(t)\dot{\sigma}(t) - \frac{\Lambda}{f} e^{-(\sigma(t)/f)} = -G\,\dot{\phi}(t)\,\langle J^0_{B-L}(t)\rangle. \quad \text{(E-o-M)}$$

We suppose that $\dot{\phi} > 0$ at early times. Assuming that the RHS of (E-o-M) approaches a non-vanishing constant, $-\lambda_{\text{cosm}}$, as $t \searrow 0$, sufficiently rapidly (so that $\sigma(t) = \mathcal{O}(t^\alpha)$, $\alpha \geq 1$, as $t \searrow 0$), we find that

$$\frac{\Lambda}{f} e^{-(\sigma/f)} \to \lambda_{\text{cosm}}\,, \text{ as } t \to 0.$$

Thus, in this modified theory, Radiation & Matter may contribute to ρ and p, and σ could also serve as an inflaton!

For very large times, the right side of (E-o-M) can be expected to tend to 0 more rapidly than t^{-2}, so that the attractor solution $\sigma^{(0)}$ will take over and produce *Dark Energy*, with $\delta \approx \frac{1}{3}$, as explained above!

Note that we have shifted the problem of exhibiting "slow rolling" of σ to understanding the origin & role of the field ϕ and showing that it satisfies an adequate "slow-roll" condition at very early times.]

The Fate of the Universe

There are other possibilities to enforce "slow rolling" of the field σ, ranging from introducing another pseudo-scalar axion to introducing not only an axion but also a new gauge field. They all look more complicated and more contrived than the ones proposed above.

Whatever the exact form of the theory, our ansatz for a theory of Dark Matter and Dark Energy leads to the following

Scenario for the evolution of the Universe:

While "slow roll" of the "quintessence field" σ (and of ϕ, in case this field appears in the theory, too) allows for *Radiation, Visible-* and *Dark Matter* to exist, *Dark Energy* will *dominate*, as $t \to \infty$. The large-time *classical dynamics* of Dark Energy appears to be well described by the solution $\sigma^{(0)}$ of the equation of motion (26).

Most regrettably, though, we do not have a concrete idea about what a *"Quantum Theory of Dark Energy"* will look like!

To conclude, it appears safe to predict (on the basis of this theory) that the large-time fate of the Universe will be very boring!

Appendix: (Pitfalls of) Fuzzy Dark Matter

Peccei and Quinn proposed that the strong CP problem could be solved by introducing an axion into Standard Model of PP. Furthermore, String Theory predicts a plethora of axions, (e.g., a "model-independent axion"). Cosmic magnetic fields may be generated by coupling an axion to the instanton density of the em field/anomalous axial lepton current, as discussed above; (see (18), (22)).

One may thus expect that axions may play an important role in cosmology. For example, axions may also represent candidates for (Fuzzy) Dark Matter. However, so far, no direct detection of Dark Matter particles!

We introduce an axion field ϕ (e.g., $\phi = f^{-1}\theta$, θ as above, $f =$ "axion decay constant"), with an action given, e.g., by

$$\boxed{S(\phi) = \frac{1}{2}\int a(t)^3 d^4x \big[f^2\, \partial_\nu\phi\, \partial^\nu\phi - \frac{1}{2}\lambda^{-4}(1 - cos\phi)\big].} \qquad (33)$$

Mass of axion $m \approx \frac{1}{4f\lambda^2}$. For FDM, $m \sim 10^{-22} - 10^{-21}$ eV. For NR velocities of order $10 km\, sec^{-1}$, one finds a de Broglie wave length of order $2 - 20\, kpc$. $\rightarrow \phi$ might not produce "small-scale" structure (?)

Axions and Fuzzy Dark Matter

To test this idea, we introduce gravitational self-interaction of axions. Because of the very small value of Newton's constant κ, we may pass to a Mean-Field description (particle creation and annihilation neglected; field ϕ approaches a classical field) and describe gravitational interactions by Newtonian gravity:

$$f\phi = \sqrt{\frac{1}{2m}}\left(\psi\, e^{-imt} + \psi^*\, e^{imt}\right).$$

The Equation of Motion for ψ reduces to the Hartree Equation:

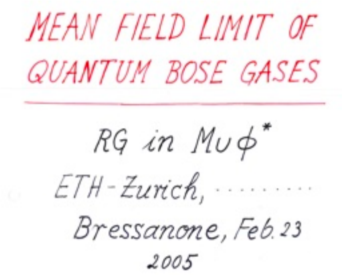
$$\begin{aligned} i\dot{\psi}(\vec{x},t) \quad &+ \quad i\frac{3}{2}H(t)\psi(\vec{x},t) = \\ &= \quad (h\psi)(\vec{x},t) - \kappa\int a(t)^3 \mathrm{d}^3y\,|\psi(\vec{y},t)|^2\frac{1}{|\vec{y}-\vec{x}|}\psi(\vec{x},t), \end{aligned}$$

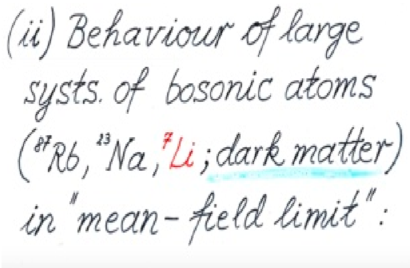
where

$$h := \sqrt{m^2 - a(t)^{-2}\Delta} - m, \quad \text{or} \quad h := -\frac{1}{2m\,a(t)^2}\Delta \tag{34}$$

Hartree Equation for Fuzzy Dark Matter

For purposes of structure formation on galactic scales, we may neglect the term $\propto H(t)$ in (34), (which causes spreading/damping), i.e., we may set $H = 0$. The Hartree Eq. (34) has been studied – analytically and numerically – for both choices of h: $h = \sqrt{m^2 - \Delta} - m$, and $h = -\frac{1}{2m}\Delta$. Here are slides of a talk I presented in 2005:





* Abou Salem, Aschbacher, JF, Hepp, Jonsson, Knowles, Lenzmann, Schlein, Schwarz, Sigal, Tsai, Yau; and others.

Hartree Solitons

Summary of some results:

- Hartree equation is a Hamiltonian equation of motion on phase space $\Gamma = \mathcal{H}_1(\mathbb{R}^3)$, with Poisson brackets

$$\{\psi^{\#}(\vec{x}), \psi^{\#}(\vec{y})\} = 0, \qquad \{\psi(\vec{x}), \psi^{*}(\vec{y})\} = i\delta(\vec{x} - \vec{y}),$$

  and Hamilton function (energy functional)

$$\mathcal{H}(\psi^*, \psi) = \int [|\vec{\nabla}\psi|^2 - \frac{\kappa}{2}|\psi|^2 \Phi * |\psi|^2], \tag{35}$$

  where $\Phi(\vec{x}) = |\vec{x}|^{-1}$ is the Newtonian gravitational potential.

- Existence of inertially moving solitons: A variational problem solved by E. Lenzmann and myself; numerical work by W. Aschbacher:

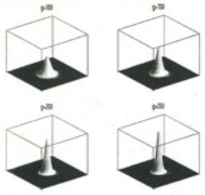
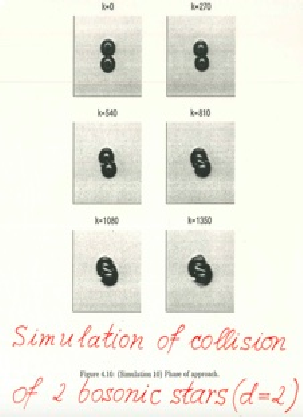
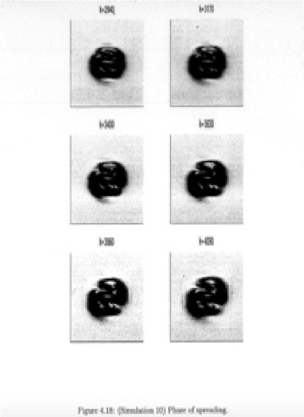


Figure 4.3: Minimizer density as a function of the Hartree coupling g

Colliding Solitons

- Equations of motion for multi-soliton configurations: Results due to Tsai, Yau, Lenzmann, J.F. ... ;(see blackboard). Numerical work by W. Aschbacher:





simulation in 2D; $k =$ time step

Gravitational Collapse of Hartee Solitons

Here we consider solitons that are minimisers of the Hamilton (energy) functional introduced in (35). Let $E_\kappa^H(N)$ denote the minimum of $\mathcal{H}(\psi^*, \psi)$ with the constraint that the conserved quantity $\|\psi|_2^2 = N$; and let $E_\kappa^Q(N)$ denote the ground-state energy of the corresponding quantum-mechanical N-body Hamilton operator, with $h := \sqrt{m^2 - \Delta} - m$, in (34). Lieb and Yau have shown that there exists a constant ν_c, with $0 < \nu_c < \infty$, such that

1. For $\nu := \kappa \cdot N < \nu_c$ fixed,
$$\lim_{N\to\infty}\big(E_\kappa^Q(N)/E_\kappa^H(N)\big) = 1$$
2. If $\nu > \nu_c$ then
$$\lim_{N\to\infty} E_\kappa^Q(N) = -\infty.$$
3. Consider a spherically symmetric initial configuration, ψ_0, with the properties that $\Delta\psi_0$ and $|\vec{x}|^2\psi_0$ are square-integrable.

Gravitational Collapse – ctd.

Assuming that

$$E_0 := \mathcal{H}(\psi_0^*, \psi_0)) + m\|\psi_0\|_2^2 < 0$$

one can show that the solution $\psi(\vec{x}, t)$ of the Hartree Eq. (34) blows up at a finite time τ, in the sense that

$$\lim_{t \nearrow \tau} \underbrace{\|\vec{\nabla}\psi(\cdot, t)\|^2}_{\text{NR kin. energy of } \psi(\cdot, t)} = \infty$$

Proof: Consider

$$Q(t) := \langle \vec{x}\,\psi_t, \cdot\sqrt{m^2 - \Delta}\,(\vec{x}\,\psi_t)\rangle > 0, \qquad \psi_t := \psi(\cdot, t)$$

Using the *virial theorem* and *Newton's theorem*, one finds that

$$0 < Q(t) \leq 2E_0 t^2 + at + b$$

Since $E_0 < 0$, R.S. becomes < 0, for t larger than some $\tau < \infty$! Def. of $Q(t)$ then shows that support of ψ_t shrinks to a point, as $t \nearrow \tau < \infty$!

Moral of Story

Gravitational blow-up for Hartree-Fock Eq. of relat. fermionic matter if $\|\psi_0\|_2^2 > \mathcal{O}(\kappa^{-(3/2)})$ ($\nearrow$ F-Lenzmann) $\rightarrow$ dynamical approach to *Chandrasekhar limit*!

Back to bosonic matter: If

$$\|\psi_0\|^2 < \mathcal{O}(\kappa^{-1}) \tag{36}$$

then Hartree Eq. (34) has global solutions. Formation of solitary waves: $\nearrow$ Terry Tao (?) – For the relativistic kinetic energy, $h = \sqrt{m^2 - \Delta}$, in (34), the number of axions in a stable soliton turns out to be bounded by const.κ^{-1} $\Rightarrow$ solitons far too small to represent galactic cores. But galactic core could be a "dilute gas" of solitons (= axion stars) with NR kinetic energy. (Yet, if the number of axions exceeds $\mathcal{O}(\kappa^{-1})$ then must, in general, expect gravitational instabilities $\rightarrow$ Formation of gases of black holes? $\rightarrow$ HOTW analysis presumably *inadequate!*)

For this model of Fuzzy DM to be viable new short-distance physics would probably have to appear already in "tiny" dense configs. of axions: Particle production- and annihilation processes ($\rightarrow$ radiation pressure); coupling of axions to other forms of DM and DR (?).

7. Conclusions?

For an outsider like myself, theoretical cosmology – in contrast to observational, phenomenological and computational cosmology – does not look like a firmly established theoretical science, yet. My impression is that the following features tend to drive one into serious difficulties:

1. The equations describing the evolution of Visible and Dark Matter and Dark Energy are highly non-linear. They might exhibit instabilities, most obviously gravitational instabilities and gauge-field dynamos, which we do not know how to treat properly, yet.
2. In every serious analysis of the dynamical evolution of the Universe one faces the problem that all forms of matter and energy are quantum-mechanical, but all gravitational degrees of freedom (g and $\sigma, \dots$) are treated classically. While there may be various self-consistent ways (e.g., semi-classical approx.) of dealing with this basic problem, it is deeply disturbing – in just about any serious study of cosmology – that we still do not know how to combine Quantum Theory with a Relativistic Theory of Gravitation.
3. On the positive side, we have made a case for the existence of extra dimensions and of an additional gravitational degree of freedom, in the form of the field σ, accounting for Dark Energy.

"Survivre et Vivre" – 54 years later

Let me conclude in French with a quote that concerns matters considerably more important than anything I've said, so far.

… depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l'espèce humaine, et même de la vie tout court, menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d'armements. …

!

Alexandre Grothendieck

Militons pour la paix!